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Non-Hamiltonian Perturbations of Integrable Systems

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Running head: Dissipative Perturbation of Hamiltonians

Abstract

We study general, non-Hamiltonian perturbations of integrable systems with two degrees of freedom. The approach is to consider asymptotic behavior of solutions near a resonant manifold, parameterized by the energy. We use action-angle variables, and averaging on a fast and an intermediate time scale, along with both canonical and non-canonical transformations of variables.

A suitable generalization of the Poincaré-Birkhoff theorem is proved, namely the existence of pairs of quasi-preserved periodic solutions, alternatingly of elliptic and hyperbolic type. These solutions appear as points on a reduced representation of the resonant manifold, coordinatized by energy and a slow phase. Near hyperbolic points, there exist open escape sets of initial data on the resonant manifold which lead to solutions leaving in finite time a given, perturbation-independent neighborhood of the manifold. Near elliptic points, we prove the existence of invariant manifolds of solutions, asymptotically stable in the large, of three types: limit cycles, two-dimensional tori and three-dimensional tori. Applications to the presence of resonances in the solar system are discussed.

1. Introduction

The motivation for this work is the study of long periods in the solar system (Arnold, 1978; Moser, 1978), on the one hand, and of slow motions of localized, coherent vortex structures in the atmosphere and ocean (Malanotte Rizzoli, 1982), on the other. In both cases, the main forces acting are conservative and the classical approach in either case has been the investigation of purely Hamiltonian perturbations of a completely integrable system.

Such a system of $2n$ pairwise conjugate variables is characterized by a Hamiltonian H^o which depends on the other $n-1$ integrals of the system, and is most conveniently represented in terms of action variables.

$$H^o = H^o(J_1, \ldots, J_n) .$$

An arbitary conservative perturbation of this system preserves only the energy, but not the other integrals of motion. Thus the perturbed system is characterized by the Hamiltonian

$$H = H^o + \varepsilon H^1(J_1, \ldots, J_n, \theta_1, \ldots, \theta_n) ,$$

θ_i being the angle variable associated with the action J_i (Arnold, 1978; Goldstein, 1980).

The formalism above was originally developed in order to study orbital variations in the solar system, and we shall concentrate on this problem in the present paper. Applications of our approach to the stability of localized coherent structures in geophysical flows (Wolansky, 1985, Chapters 3 and 4) will be given in a separate publication.

For the solar system, H^o is the sum of Hamiltonians of several independent

Kepler oscillators, whose evolution is completely integrable. The perturbation parameter ε is a measure of the mass ratio between planets and Sun, with $\varepsilon \cong 10^{-3}$, and H^1 contributes the gravitational interaction between planets.

The approximate solution of this problem as an initial-value problem, by either numerical or perturbative methods, will lead to errors $O(1)$ in time $O(1/\varepsilon)$. For the solar system such errors would mean a noticeable deformation of orbits in a time of thousands of years, a very short time when compared to the age of the system. Thus stability arguments for the solar system require asymptotic results, the most powerful ones being given by Kolmogorov-Arnold-Moser (KAM) theory.

The main conclusion of KAM theory can be formulated as follows: Under a nondegeneracy condition on H^o, for ε small enough and H^1 smooth enough, most of the unperturbed phase-space flow's invariant tori will survive the perturbation. On the surviving invariant tori the flow is quasi-periodic, with irrational rotation number which is poorly approximated by rationals (e.g., Kolmogorov, 1954).

The conclusion above does not guarantee stability, unless we are dealing with the case of two degrees of freedom, in which the invariant tori separate portions of phase space where the motion is aperiodic (Lichtenberg and Lieberman, 1983). In addition, the theory fails on those invariant tori for which the rotation number is rational (resonant tori) or nearly rational. It seems, however, that the solar system is characterized by many resonance or near-resonance relations (Molchanov, 1969). This observation does not appear to be consistent with applying the above conclusion of KAM theory to the system (see also Duriez, 1982).

The obvious importance of periodic (resonant) trajectories served as a strong motivation for Poincaré, Birkhoff and others (e.g., Siegel and Moser, 1971, and references therein) to investigate special periodic solutions to the n-body problem and their stability. The survival of periodic solutions under Hamiltonian perturbations for the case of two degrees of freedom is the consequence of the Poincaré-Birkhoff theorem which was extended, under certain conditions, to three or more degrees of freedom (Arnold, 1978, App. 9). In the case of two degrees of freedom, the theorem yields the survival of at least two periodic solutions on each resonant torus, of alternating elliptic and hyperbolic type. Each elliptic-type solution is surrounded by invariant KAM tori, which guarantee its stability. No such stability result is available in the case of three or more degrees of freedom.

Even if we restrict our attention to two degrees of freedom, there is no apparent reason why the physical system should prefer to stay at or near a resonance, rather than on a KAM torus far from resonance. The existence of many resonance relations in the solar system may indicate, as suggested by Goldreich (1965), that the purely Hamiltonian formalism is not valid over a very long time scale, over which nonconservative effects, such as tidal dissipation and radiation pressure, could become significant.

Motivated by such considerations, we study in this paper a system of the form:

$$\frac{dJ_i}{dt} = \epsilon\, f_i(J_1,\ldots,J_n,\theta_1,\ldots,\theta_n) \quad , \tag{1.1a}$$

$$i = 1,\ldots,n,$$

$$\frac{d\theta_i}{dt} = \frac{\partial H^o}{\partial J_i} + \epsilon\, g_i(J_1,\ldots J_n,\theta_1,\ldots \theta_n), \tag{1.1b}$$

where f_i, g_i are 2π-periodic in the angle variables, but otherwise arbitary. This formalism models the case of non-Hamiltonian perturbations of integrable systems. In practical applications one has to assume:

$$f_i = -\frac{\partial}{\partial\theta_i} H^1 + \delta \bar{f}_i \ . \ g_i = \frac{\partial}{\partial J_i} H^1 + \delta \bar{g}_i \qquad (1.2a.b)$$

where $\bar{g}_i$, $\bar{f}_i$ are the non-Hamiltonian parts and δ is assumed to be small, but independent of ε. One may expect that, in general, the invariant tori will disintegrate under such a perturbation. Murdock (1975, 1976) proved certain nonexistence theorems for invariant tori and indicated that all the surviving tori will be confined to a small part of the phase space.

Our approach to the problem is based on a different point of view. Instead of looking for invariant tori, we investigate conditions under which solutions will stay on, near or escape far from a certain resonant manifold, which is the union of all resonant tori admitting a certain resonance relation. In this way, we expect to obtain domains of atraction for certain resonance relations, independently of ε. In particular, the analysis indicates generalizations, in an appropriate sense, of the Poincaré-Birkhoff theorem.

Another aspect of the analysis provides a generalization of the theory of adiabatic invariants. The usual theory concerns only (1.1) of Hamiltonian type and depending on a slow time parameter. Applications of this theory to Celestial Mechanics have been given by Henrard (1982).

In the present paper we deal, essentially, with the case of two degrees of freedom. Some of the results can be extended to a larger number of degrees of freedom and this will be the main topic of a separate paper.

In Sect. 2 we perform a noncanonical transformation of system (1.1) and average with respect to the fast phase in the neighborhood of the resonant manifold, to study changes on the $\varepsilon^{1/2}$-time scale. This yields the system in a standard form (2.8), which highlights the fact that the averaged perturbation is conservative to leading order. Section 3 deals with solutions in the neighborhood of hyperbolic-type points. The main result of this section, proved in Theorem 1, is the existence of a codimension-one family of near-resonant solutions. In Sect. 4 we derive Theorem 2, which shows the existence of an open set of solutions escaping the resonant manifold.

In Sect. 5 we concentrate on the neighborhood of elliptic-type points. There we use the conservative nature of the averaged perturbation to leading order, and apply a canonical transformation of the averaged system to new action-angle variables. This enables us to define "slow averaging" with respect to the $\varepsilon^{1/2}$-time scale and reduce the system to a pair of equations on the ε-time scale, decoupled from the other two variables to leading order. As a result we obtain Theorem 3 and a corollary, which give conditions for resonant trapping. In particular, we conclude the generic existence of three types of attractors: periodic solutions, two-dimensional tori and three-dimensional tori in certain domains of the resonant manifold. The last type of attractor is characteristic of non-Hamiltonian systems only.

To complete the analysis of solution behavior near elliptic-type points (Theorem 3) and near hyperbolic-type points (Theorem 1), we give Theorem 4 on behavior in the neighborhood of the separatrix between an elliptic and a hyperbolic branch of nearly-periodic solutions. Sect. 5 concludes with remarks on the characterization of purely dissipative perturbations within our general framework of non-Hamiltonian perturbations. In Sect. 6 we discuss the above results in connection with the stability problem of the solar system, and the actual computation of long periods. Three appendices give details of proofs.

2. Reduction and Averaging Method

Consider system (1.1) with two degrees of freedom ($n=2$), where $H^o = H^o(J_1,J_2)$ is the unperturbed Hamiltonian, given in terms of action variables, and the perturbations $\{f_i\}$, $\{g_i\}$ are 2π-periodic in the corresponding angle variables. The perturbation parameter ε is assumed to be small, f_i and g_i are C^k functions, $k \geqslant 4$, in all variables, and $H^o \in C^{k+1}$.

Substituting $\varepsilon = 0$ in (1.1) leads to the integrable system

$$\frac{dJ_i}{dt} = 0 \quad , \quad \frac{d\theta_i}{dt} = \omega_i \ , \ i = 1,\ 2, \tag{2.1}$$

where $\omega_i := H^o_{J_i}$, and we use ":=" to indicate a defining identity. Thus, J_i, $i = 1,\ 2$ are integrals of the motion and the flow is either periodic (ω_1/ω_2 = rational) or else quasi-periodic.

Given a certain resonance relation p/q where p and q are mutually prime, we define $W = W(J_1,J_2)$ by

$$W := (p^2+q^2)^{-1/2}\ (p\omega_2-q\omega_1)\ . \tag{2.2}$$

The resonant manifold $M_{p,q}$ associated with the above relation is a codimension-one submanifold in phase space given by:

$$M_{p,q} := \left\{J_i,\ \theta_i\ :\ W=0,\ \theta_1,\ \theta_2 \in T^2\right\}\ ,$$

where $T^2 = S^1 \times S^1$ is the standard two-torus.

We are interested in the characterization of those solutions of (1.1) which are nearly resonant during the system's evolution, i.e.,

$$|W| \ll 1.$$

As will be seen later on, a natural requirement for a nearly phase-locked solution is:

$$W = O\,(\sqrt{\varepsilon})\ .$$

Assumption 2.1. The inequality

$$\det \left(\frac{\partial\,(H^{o}, W)}{\partial(J_1, J_2)}\right) \neq 0$$

holds in a neighborhood of $W = 0$.

The above assumption corresponds to the condition that the unperturbed energy surfaces H^{o} = *const.* intersect transversally the resonant manifold $M_{p,q}$. Under Assumption 2.1 we can replace, in the neighborhood of $M_{p,q}$, the action variables by the pair of independent functions (W, E),

$$(J_1, J_2) \rightarrow (W, E), \tag{2.3a,b}$$

where $E = H^{o}$. We introduce also the following transformation in angle space:

$$\psi = (p^2 + q^2)^{-1/2} (p\theta_1 + q\theta_2), \tag{2.3c}$$

$$\phi = (p^2 + q^2)^{-1/2} (q\theta_1 - p\theta_2). \tag{2.3d}$$

Clearly ψ is a fast and θ is a slow variable near $M_{p,q}$.

With the change of action and angle variables above, system (1.1) becomes

$$\dot{W} = \varepsilon F^1 \, (W, E, \phi, \psi) \ , \tag{2.4a}$$

$$\dot{E} = \varepsilon F^2 \, (W, E, \phi, \psi) \ , \tag{2.4b}$$

$$\dot{\phi} = W + \varepsilon G^2 (W, E, \phi, \psi) \ , \tag{2.4c}$$

$$\dot{\psi} = W^1 \, (W, E) + \varepsilon G^2 \, (W, E, \phi, \psi); \tag{2.4d}$$

here

$$F^1 := f_1 \frac{\partial}{\partial J_1} W + f_2 \frac{\partial}{\partial J_2} W \ , \tag{2.5a}$$

$$F^2 := \omega_1 f_1 + \omega_2 f_2 \ , \tag{2.5b}$$

$$G^1 := (p^2 + q^2)^{-\frac{1}{2}} \left[qg_1 - pg_2\right] , \tag{2.5c}$$

$$G^2 := (p^2 + q^2)^{-\frac{1}{2}} \left[pg_1 + qg_2\right] , \tag{2.5d}$$

and

$$W^1 := (p^2 + q^2)^{-\frac{1}{2}} (p\omega_1 + q\omega_2) . \tag{2.5e}$$

All the functions above are expressed in terms of the new set of variables.

On the resonant manifold

$$W^1 = (\omega_1^2 + \omega_2^2)^{\frac{1}{2}} \neq 0 .$$

By Assumption 2.1 and without loss of generality, we may also assume that

$$W^1 \geqslant \mu > 0$$

in a neighborhood of the resonant manifold, on which the transformation $\begin{Bmatrix} J_i \end{Bmatrix} \to \begin{Bmatrix} W,E \end{Bmatrix}$ is invertible. Notice that for $\varepsilon = 0$, $W = 0$, ϕ is a constant of system (2.4), while

$$\dot{\psi} = W^1 (0,E) = \text{const} \geqslant \mu ,$$

which makes our earlier remark on the new angle variables more precise.

Given an arbitrary function $g(\theta_1, \theta_2)$ on T^2 we define its average $\langle g\rangle = \langle g\rangle (\theta_1, \theta_2)$ on the trajectories of the unperturbed system by

$$\langle g\rangle := \frac{1}{T} \int_0^T g(\theta_1(t), \theta_2(t))dt , \tag{2.6}$$

where $\theta_i(t)$ are solutions of (2.1) and $T = T(E)$ is the period of the unperturbed solution on $W = 0$.

Lemma 2.1. *Let $g(\theta_1,\theta_2)$ be a function on T^2. Then under the transformation* (2.3), *g is given as a function on the torus $\bar{T}^2$:*

$$\bar{T}^2 := \bar{S}^1 \times \bar{S}^1 ,$$

where

$$\bar{S}^1 := \mathbb{R} \bmod 2\pi(p^2 + q^2)^{\frac{1}{2}} ,$$

while $\langle g\rangle = \langle g\rangle(\phi)$ is a $2\pi(p^2 + q^2)^{-\frac{1}{2}}$ - periodic function of ϕ.

Proof. It suffices to consider a Fourier component of g,

$$\exp\left\{i(k\theta_1 + \ell\theta_2)\right\} ,$$

with k and ℓ integers. In terms of ψ and θ, the above component is converted into:

$$\exp\left\{i\left[(pk + q\ell)\psi + (p\ell - qk)\phi\right](p^2 + q^2)^{-\frac{1}{2}}\right\}$$

As p and q are mutually prime, there exist k, ℓ, k', ℓ' such that

$$pk + g\ell = \pm 1,\ p\ell' - qk' = \pm 1 .$$

Hence g does have a minimal period of $2\pi(p^2 + q^2)^{\frac{1}{2}}$ in both ψ and θ.

The average of g over the unperturbed period of θ_1 and θ_2 corresponds to an average with respect to ψ over its minimal period, keeping ϕ fixed. Thus, the only harmonic components left by the averaging are given by

$$pk + q\ell = 0$$

or

$$k = -\gamma q,\ \ell = \gamma p\ ,\ \gamma = 0,\ \pm 1,\ \pm 2\ \ldots\ .$$

Hence

$$\langle g\rangle = -\sum_{\gamma} g_{-\gamma q,\gamma p} \exp i\left\{\gamma(p^2 + q^2)^{\frac{1}{2}}\ \phi\right\}$$

and $\langle g\rangle$ has indeed the minimal period $2\pi(p^2 + q^2)^{-\frac{1}{2}}$. *QED*

The geometric relation between the standard, 2π-periodic torus T^2 of the original variables θ_1, θ_2, the resonantly covering, $2\pi(p^2+q^2)^{\frac{1}{2}}$- periodic torus $\bar{T}^2$ of the rotated, fast/slow variables ψ, ϕ, and the minimal period $2\pi(p^2+q^2)^{-1/2}$ in ϕ after averaging with respect to ψ is shown in Fig. 1.

[Fig. 1 near here, please]

The representation (2.4) of system (1.1) is valid on an otherwise arbitrary open subset of phase space for which Assumption 2.1 holds. However, we are interested in characterizing resonantly locked solutions for which ϕ is still a slow variable (or, equivalently, $| W | \ll 1$), presumably near $M_{p,q}$. For this reason, we want to apply an averaging procedure to (2.4) by a transformation

$$W \to W + \epsilon u_1\ (W,\ E,\ \phi,\ \psi)\ , \tag{2.7a}$$

$$E \to E + \epsilon u_2\ (W,\ E,\ \phi,\ \psi)\ , \tag{2.7b}$$

$$\phi \to \phi + \epsilon u_3\ (W,\ E,\ \phi,\ \psi)\ , \tag{2.7c}$$

where we require u_i, $i = 1, 2, 3$, to be uniformly bounded in the domain under consideration and periodic in ϕ and ψ. To construct the functions u_1, u_2, u_3 explicitly we need some minor machinery.

Definition 2.1. Let the operator $\mathcal{K}$ be defined on the function space $\mathbb{L}$,

$$\mathbb{L} := \left\{h:\ h = h(W,\ E,\ \phi,\ \psi)\ \epsilon C^1,\ \langle h\rangle\ (W,\ E,\ \phi) = 0\right\}\ ,$$

by

$$\mathcal{K}\cdot h := \int^{\psi} h(W,\ E,\ \phi,\ s)ds \text{ and } \langle\mathcal{K}\cdot h\rangle = 0.$$

From the definition and Lemma 2.1 we immediately obtain the following:

Proposition. *$\mathcal{K}\cdot h$ is $2\pi(p^2 + q^2)^{\frac{1}{2}}$ periodic in both ϕ and ψ, and $\mathcal{K}^n$ is well* *defined for every integer* $n \geq 1$ *and* $h \in \mathbf{I}$.

This proposition permits us to write

$$u_1 = \frac{1}{W^1(W,E)} \left[\mathcal{K}\cdot(F^1 - \langle F^1\rangle) - \frac{W}{W^1(W,E)} \mathcal{K}^2\cdot(F^1 - \langle F^1\rangle)_\phi \right] ,$$

$$u_2 = \frac{1}{W^1(W,E)} \mathcal{K}\cdot\left[F^2 - \langle F^2\rangle\right] ,$$

$$u_3 = \frac{1}{W^1(W,E)} \mathcal{K}\cdot\left[u_1 + G^1 - \langle G^1\rangle\right] .$$

Inserting now the transformation (2.7) so defined in (2.4), we conclude:

Lemma 2.2. *System (2.4) is equivalent near* $M_{p,q}$ *to*

$$\dot{W} = \varepsilon\, \bar{F}^1\,(W,\ E,\ \phi) + O(\varepsilon\, W^2) + O(\varepsilon^2)\ , \tag{2.8a}$$

$$\dot{E} = \varepsilon\, \bar{F}^2\,(W,\ E,\ \phi) + O(\varepsilon^2) + O(\varepsilon W)\ , \tag{2.8b}$$

$$\dot{\phi} = W + \varepsilon\, \bar{G}^1(W,\ E,\ \phi) + O(\varepsilon^2) + O(\varepsilon W), \tag{2.8c}$$

$$\dot{\psi} = W^1(W,E) + O(\varepsilon)\ , \tag{2.8d}$$

where overbars designate the ψ*-averaged variables*

$$\bar{F}^i = \langle F^i\rangle \in C^k,\ i = 1,\ 2, \qquad \bar{G}^1 = \langle G^1\rangle \in C^k\ ,$$

and the higher-order terms are C^{k-1}, *well defined in* ϕ *and* ψ *over* $\bar{T}^2$.

Since we are interested in solutions in the neighborhood of the resonant manifold, we transform (2.8) by

$$\bar{W} = \varepsilon^{-\frac{1}{2}}\, W\ ,\ \tau = \varepsilon^{\frac{1}{2}}\, t\ ,$$

to

$$\frac{d}{d\tau}\ \bar{W} = \hat{F}^1\,(E,\ \phi) + \sqrt{\varepsilon}\ \hat{F}^1_W\,(E,\ \phi)\ \bar{W} + O(\varepsilon)\ , \tag{2.9a}$$

$$\frac{d}{d\tau}\,\phi \;=\bar{W} + \sqrt{\varepsilon}\;\hat{G}^1\;(E,\;\phi) + 0(\varepsilon)\;, \tag{2.9b}$$

$$\frac{d}{d\tau}\,E \;= \sqrt{\varepsilon}\;\hat{F}^2\;(E,\;\phi) + 0(\varepsilon)\;, \tag{2.9c}$$

$$\frac{d}{d\tau}\,\psi \;= \frac{1}{\sqrt{\varepsilon}}\;W^1\;(0,E) + 0(1)\;; \tag{2.9d}$$

here

$$\hat{F}^i\;(E,\;\phi) := \bar{F}^i\;(0,\;E,\;\phi)\;,$$

$$\hat{G}^i\;(E,\;\phi) := \bar{G}^i\;(0,\;E,\;\phi)\;,$$

$$\hat{F}^1_W := \frac{\partial}{\partial W}\,\bar{F}^1\;(W,\;E,\;\phi)\Big|_{W=0}\;.$$

If we substitute now $\varepsilon = 0$ in (2.9) we get, on the $\sqrt{\varepsilon}$-time scale:

$$\frac{d}{d\tau}\,\bar{W} = \hat{F}^1\;(E,\;\phi)\;,$$

$$\frac{d}{d\tau}\,\phi = \bar{W}\;,$$

$$\frac{d}{d\tau}\,E = 0\;,$$

which can be written as

$$\frac{d^2}{d\tau^2}\;\phi = \hat{F}^1\;(E,\;\phi)\;,\quad E = const.$$

3. Solutions near the Hyperbolic Branch

Every constant value of E corresponds, due to Assumption 2.1, to a single resonant torus in phase space, on which the fixed p/q resonance relation holds. Given $E = E^o$, we consider the set of roots ϕ_i, $i = 1,\ldots,n$, of:

$$\hat{F}^1 \; (E^o, \; (\phi_i(E^o)) = 0 \; , \tag{3.1a}$$

subject to

$$\frac{\partial}{\partial\phi} \hat{F}^1\Big|_{E=E^o, \phi=\phi_i} \neq 0 \; . \tag{3.1b}$$

Definition 3.1. Given $E = E^o$, we call each root of (3.1) a *quasi-preserved point of elliptic* or *hyperbolic* type if $\frac{-\partial}{\partial\phi} \hat{F}^1\Big|_{\phi=\phi_i} < 0$ or $\frac{\partial}{\partial\phi} \hat{F}^1\Big|_{\phi=\phi_i} > 0$.

From the periodicity of $\hat{F}^1$ with respect to ϕ and the implicit function theorem, we easily conclude:

Lemma 3.1. *There exist, generically, an even number of quasi-preserved points on any resonant torus in the resonant manifold. Exactly half of these are of elliptic type and the other half are of hyperbolic type. Any of these points can be extended to local elliptic/hyperbolic branches*

$$\phi_i = \phi_i(E)$$

in the neighborhood of $(E^o, \; \phi_i(E^o))$.

Remark. The quasi-preserved points defined above turn out to be, in the case of Hamiltonian perturbations, an approximation to those periodic solutions which are preserved according to the Poincaré-Birkhoff theorem. In fact, for Hamiltonian perturbations we have

$$\int_0^{\bar{\phi}} \hat{F}^1 \; (E,\phi)\; d\phi = 0 \;, \quad \bar{\phi} = 2\pi(p^2+q^2)^{-\frac{1}{2}} \;,$$

and thus at least two quasi-preserved points always exist. In the case of general perturbations, however, the even number of points given by Lemma 3.1 may be zero.

In Fig. 2a we show an illustrative example of a contour of $\hat{F}^1 = 0$ on the resonant manifold. A point (E, ϕ) on a branch of quasi-preserved solutions $\phi = \phi(E)$ at which both $\hat{F}^1 = 0$ (cf. Eq. (3.1a)) and $\hat{F}^2(E,\phi(E)) = 0$ is a stationary point of the reduced phase flow. Such a point corresponds to a unique periodic solution of Eqs. (1.1), as will be shown in Sect. 5 and Appendix C. Fig. 2b illustrates the stability properties of elliptic and hyperbolic stationary points (compare also Eq. (3.2) below).

[Fig. 2 near here, please]

Let now $\phi_i = \phi_i^o(E)$ be one of the hyperbolic branches given by Lemma 3.1. Substituting $\phi_i^o(E)$ in the energy equation (2.9c) we get to leading order,

$$\frac{dE}{d\tau} = \sqrt{\epsilon}\; \hat{F}^2 \; (E,\; \phi_i(E)) \quad . \tag{3.2}$$

Assume that

$$\frac{\partial}{\partial\phi} \hat{F}^1 \Big|_{\phi=\phi_i\;(E)} > \nu > 0 \tag{3.3}$$

for $E_1 < E < E_2$ and let $E = E^o(\tau)$ be the solution of (3.2), subject to the initial condition

$$E^o(0) = E_o \varepsilon (E_1,E_2) \;.$$

Let $T^{E_o,\nu}$ be the time in which $E^o(\tau)$ leaves the (E_1,E_2) interval. Thus

$$E^o(\tau) \varepsilon \; (E_1,\; E_2) \;\; , \;\; 0<\tau< T^{E_o,\nu} \;,$$

$$E^o(T^{E_o,\nu}) = E_1 \text{ or } E_2 \ .$$

If $\hat{F}^2(E,\phi_i(E)) = 0$ for some $E\ \varepsilon(E_1,E_2)$, we define $T_{\bullet}^{E,\nu} = \infty$. Assume further

$$J(\hat{F}^1, \hat{F}^2) = \frac{\partial}{\partial E}\hat{F}^2 \frac{\partial}{\partial \phi}\hat{F}^1 - \frac{\partial}{\partial \phi}\hat{F}^2 \frac{\partial}{\partial E}\hat{F}^1 < -\bar{\nu} < 0 \qquad (3.4)$$

in $(E,\ \phi_i(E))$, $E\ \varepsilon(E_1,E_2)$, i.e., that $\hat{F}^2(E,\phi_i(E))$ is monotonically decreasing in E along $\hat{F}^1 = 0$ in this interval. Notice that inequalities (3.3, 3.4) will both hold for time $t = O(1/\varepsilon)$ or $\tau = O(1/\sqrt{\varepsilon})$.

Given $\delta > 0$ and E as above, define an open neighborhood U_E^δ in the action space by

$$U^\delta{}_E = \left\{J_1,\ J_2\ : \left|H^o(J_1,\ J_2) - E\right| + \left|\bar{w}\right| < \delta\right\} ,$$

and a neighborhood $V^\delta{}_E$ of $\phi_i^o(E)$ by

$$V_E^\delta = \left\{\phi : \left|\phi - \phi_i^o(E)\right| < \delta\right\} .$$

Consider a three-dimensional hyperplane, transversal to the flow near the resonant manifold and given by, say, $\psi = 0$. The following theorem deals with all solutions of (1.1) which cross the transversal section $\psi = 0$ within an open set in U_E^δ at $\tau = 0$.

Theorem 1. *Assume (3.3, 3.4). Given $\delta > 0$ small enough, there exist an open set $\bar{U}^\delta \subset U_{E_o}^\delta$ in action space and a continuous map*

$$\Phi : \bar{U}^\delta \to V_{E_o}^\delta$$

such that any solution of (1.1) which admits the initial data

$$\psi = 0,\ \phi = \Phi(E,\bar{W})\ \textit{for}\ (E,\ \bar{W}) \in \bar{U}^{\delta}$$

will be confined to

$$(E(\tau),\ \bar{W}(\tau)) \in U^{\delta}_{E^{o}(\tau)} \quad , \quad \phi(\tau) \in V^{\delta}_{E^{o}(\tau)}$$

for $0 \leq \tau \leq T^{E_o,\nu}$, *where* $E^{o}(\tau)$ *is the solution of Eq.* (3.2) with $E^{o}(\tau) = E_o$.

In particular, such solutions stay in the neighborhood of the resonant manifold, at least as long as condition (3.3) holds. Notice that neither $T^{E_o,\nu}$ nor δ depend on ε, for ε small enough. Thus, if $T^{E_o,\nu} = \infty$, Theorem 1 yields a codimension-one submanifold of initial data for which the solutions will stay near the resonant manifold indefinitely. In this case, the reduced phase flow of the system is completely portrayed by the flow indicated near point 5 in Fig. 2b.

Proof of Theorem 1. Let us substitute

$$E(\tau) = E^{o}(\tau) + e(\tau) \quad ,$$

$$\phi(\tau) = \phi^{o}(\tau) + \theta(\tau) \quad ,$$

where $\phi^{o}(\tau)$ is given as $\phi^{o}(E^{o})$ by Eq. (3.1) and $E^{o}(\tau)$ by Eq. (3.2). Using Taylor expansion for $\hat{F}^1$, $\hat{F}^2$ near $E^{o}(\tau)$, $\phi^{o}(\tau)$, we get from (2.9)

$$\frac{d}{d\tau}\bar{W} = \hat{F}^1_{\phi}\,\theta + \hat{F}^1_{E}\,e + 0(\sqrt{\varepsilon}) + 0\,(\theta^2 + e^2)\ , \tag{3.5a}$$

$$\frac{d\theta}{d\tau} = \bar{W} + 0(\sqrt{\varepsilon})\ , \tag{3.5b}$$

$$\frac{de}{d\tau} = \sqrt{\varepsilon}\,(\hat{F}^2_{\phi}\,\theta + \hat{F}^2_{E}\,e) + 0(\varepsilon) + 0(\sqrt{\varepsilon}(\theta^2+e^2)). \tag{3.5c}$$

The coefficients of θ and e are slow functions of time, since $\frac{d}{d\tau}\phi^{o} = 0(\sqrt{\varepsilon})$.

The higher-order terms in the system above are periodic functions of ψ as well, with ψ given by Eq. (2.4). However, the whole system can be transformed so that $\sqrt{\varepsilon}\psi$ replace τ as an independent variable, thus yielding a closed system of three non-autonomous equations.

The transformation

$$e = \bar{e} + \sqrt{\varepsilon}\,\frac{\hat{F}^2_\phi}{\hat{F}^1_\phi}\,\bar{W}$$

is well defined and uniformly bounded according to (3.3). With it (3.5c) becomes

$$\dot{\bar{e}} = -\sqrt{\varepsilon}\,\lambda^E\,\bar{e} + O(\varepsilon) + O(\sqrt{\varepsilon}(\theta^2 + \bar{e}^{\,2})),$$

where $(\quad)^{\cdot} = \varepsilon^{-1/2}\,d(\quad)/d\psi$ and

$$\lambda^E := -\frac{1}{\hat{F}^1_\phi}\left(\hat{F}^2_E\,\hat{F}^1_\phi - \hat{F}^2_\phi\,\hat{F}^1_E\right).$$

By (3.4)

$$\lambda^E > \frac{\bar{\nu}}{\max \hat{F}^1_\phi} > \mu_1 > 0,$$

and furthermore

$$\dot{\lambda}^E = O(\sqrt{\varepsilon}).$$

Upon inserting

$$\theta = \bar{\theta} - \frac{\hat{F}^1_E}{\hat{F}^1_\phi}\,e$$

into (3.5), the term $\hat{F}^1_E\,e$ in that equation drops out, at the cost of merely changing the higher-order terms. Over all, the estimate $O(\sqrt{\varepsilon}) + O(e^2 + \theta^2)$ remains valid for the transformed Eqs. (3.5a-c).

The transformation

$$z_1 = \sqrt{\hat{F}^1_\phi}\ \theta + \bar{W} \ ,$$

$$z_2 = - \sqrt{\hat{F}^1_\phi}\ \theta + \bar{W} \ ,$$

which is real and nondegenerate, due to (3.3), leads *in fine* to the system

$$\dot{z}_1 = \lambda z_1 + h_1 \ , \tag{3.6a}$$

$$\dot{z}_2 = - \lambda z_2 + h_2 \ , \tag{3.6b}$$

$$\dot{\bar{e}} = - \sqrt{\varepsilon}\ \lambda^E e + h_3 \ ; \tag{3.6c}$$

here h_1, h_2 and h_3 are functions of z_1, z_2, $\bar{e}$ and ψ, with

$$h_i = O(\bar{e}^2 + z_1^2 + z_2^2) + O(\sqrt{\varepsilon}) \ , \ i = 1, 2 \ ,$$

$$h_3 = O(\varepsilon) + O(\sqrt{\varepsilon}(\bar{e}^2 + z_1^2 + z_2^2)) \ ,$$

while

$$\lambda := \sqrt{\hat{F}^1_\phi} > \mu_2 > 0$$

uniformly over the domain in question and

$$\dot{\lambda} = O(\sqrt{\varepsilon}) \ .$$

Thus Eqs. (3.6b,c) describe approximately the flow along the analog of the stable manifold for a quasi-preserved point $(E_o, \phi_i(E_o))$ given by (3.1).

Introduce now the operator $\mathcal{Z}$, mapping the Banach space $\mathbb{B}$ of three-component vector functions

$$(z_1(\cdot), z_2(\cdot), \bar{e}(\cdot)) \in C^0 (0, T^{E_o,\nu}) \ ,$$

into itself. Component-wise,

$$\mathcal{Z}^1 := \int_{\tau}^{T^{E_o.\nu}} \exp\left[-\int_{\tau}^{s} \lambda(w)dw\right] h_1(z_1,\ z_2(s),\ \bar{e}\ (s),s)ds\ ,$$

$$\mathcal{Z}^2 := z_2^o \exp\left[-\int_{o}^{\tau} \lambda(s)ds\right] + \int_{o}^{\tau} \exp\left[-\int_{s}^{\tau} \lambda(w)dw\ h_2\ (z_1,\ z_2(s),\ \bar{e}(s),s)ds\right.$$

$$\mathcal{Z}^3 := e^o \exp\left[-\sqrt{\varepsilon} \int_{o}^{\tau} \lambda^E(s)\ ds\right] + \int_{o}^{\tau} \exp\left[-\sqrt{\varepsilon}\int_{s}^{\tau} \lambda^E(w)\ dw\right] h_3\ (z_1,\ z_2(s),\ \bar{e}(s),s)ds\ ,$$

and $\mathcal{Z}$ depends upon the parameters $(z_2^o,\ e^o)$.

Due to the lower bounds on λ and λ^E, it is evident that $\mathcal{Z}$ maps $\mathbb{B}$ into itself, and the bound is independent of $T^{E_o.\nu}$, for every value of the parameters $(z_2^o,\ e^o)$. Furthermore, one can find $\delta_o > 0$ for which $\mathcal{Z}$ maps a cylinder

$$C^2_{\delta_o} := \left\{z_1,\ z_2,\ \bar{e}\ :\ |z_2^o| + |e^o| < \delta_o/2\right\} \tag{3.7a}$$

into the ball

$$B^3_{\delta_o} := \left\{z_1,\ z_2,\bar{e}\ :\ \|z_1\| + \|z_2\| + \|\bar{e}\| < \delta_o\right\}\ . \tag{3.7b}$$

Finally $\mathcal{Z}$ can be shown to be a contraction for a properly defined $\delta < \delta_o$ and ε small enough, due to the Lipschitz condition obtainable from the higher differentiability of h_i, $i = 1,2,3$. Thus, we get for every pair $(z_2^o,\ e^o)$ in C^2_{δ} a fixed point of $\mathcal{Z}$. Such a fixed point is clearly a solution of (3.6), which admits the initial data

$$z_2(0) = z_2^{\ o},\ \bar{e}(0) = e^o,$$

$$z_1\ (0) = \int_0^T exp\left[-\int_s^{\tau} \lambda(w)dw\right] h_1\ ds\ \ .$$

Since $\mathcal{Z}$ is a contraction on C^2_δ , its fixed points are continously dependent on the parameters (z^o_2, e^o). Thus $z_1(0)$ is a continuous function of the above pair. We can now define $U^\delta{}_E$ as $B^3{}_\delta$, in terms of $(z_1, z_2, \bar{e})$ centered at $E\epsilon(E_1, E_2)$ and $\bar{W} = 0$, $\phi = \phi^o_i(E)$ for $E_1 < E < E_2$, and $\bar{U}^\delta$, $V^\delta_{E_o}$ as appropriate open sets such that:

$$\bar{U}^\delta \times V^\delta_{E_o} \subset B^3_\delta$$

centered at $E = E_o$, $\bar{W} = 0$ and $\phi = \phi^o_i(E_o)$. The function $\Phi : \bar{U}^\delta \to V^\delta_{E_o}$ is given by the graph of $z^o_1 = z^o_1(z^o_2, e^o)$ in the above set, and Theorem 1 follows by the contraction argument. *QED*

4. Resonance Breaking and Escaping Solutions

Theorem 1 deals with two-dimensional submanifolds in the three-dimensional reduced phase space given by the Poincaré map with respect to a transversal section $\psi = 0$. Any initial data on such a submanifold will lead to a solution trapped at least for a time interval $T^{E_o,\nu}/\sqrt{\epsilon}$ near the resonant manifold.

Even more interesting is to find open sets in the reduced phase space for which any solution, initially in such a set, will be trapped permanently on the resonant manifold. Before investigating such sets, we will deal with the opposite question of resonance breaking through solutions escaping the neighborhood of $M_{p,q}$.

Definition 4.1. An open set D in the resonant manifold $M_{p,q}$ given by $(E,\phi)\epsilon D \subset \mathbb{R} \times \bar{S}^1$, will be called an *escape set* if there exist $\epsilon_o > 0$ and $\delta > 0$ real, such that for any $0 < \epsilon < \epsilon_o$, any solution of (1.1) with initial data in the set Ω,

$$\Omega := \left\{W,\ E,\ \phi\colon\ (E,\phi)\ \epsilon\ D\ \ ,\ W\epsilon\ \left[0,\delta\right]\right\}$$

will escape Ω at $W = \delta$, in finite time. The same definition applies if $W\epsilon\left[\delta,0\right]$

and $\delta < 0$.

Notice that neither δ nor D is assumed to depend on ε, as long as ε is small enough, and $\bar{S}^1$ is as in Lemma 2.1.

Theorem 2. *Let*

$$\bar{F}(E) := \frac{1}{2\pi}\,(p^2+q^2)\int_0^{2\pi(p^2+q^2)^{-1/2}} \hat{F}^1\,(E,\phi)d\phi \tag{4.1a}$$

and assume there is a $\mu > 0$ such that

$$\bar{F} < -\mu \text{ or } \bar{F} > \mu \text{ on } E\,\varepsilon(E_1,\ E_2)\ . \tag{4.1b,c}$$

Then there exists an escape set $D \subset R \times \bar{S}^1$, whose projection on the E - coordinate lies in $(E_1,\ E_2)$.

Before proving Theorem 2, we state Lemma 4.1 and Corollary 4.1.

Lemma 4.1. *Assume $\hat{F}(E) > \mu$ on $(E_1,\ E_2)$. Then there exist, for ε small enough, $b > 0$ and $\delta > 0$ such that any solution with initial data in the set*

$$\bar{\Omega}^\varepsilon = \left\{W,\ E,\ \phi\ :\ \ b\sqrt{\varepsilon} \leq W \leq \delta,\ E_1 \leq E \leq E_2,\ \phi\ \varepsilon\ \bar{S}^1\right\}$$

will escape this set at $W = \delta$. The same result applies if $\hat{F}(E) < -\mu$, $b < 0$ and $\delta < 0$ for $\delta \leq W \leq b\sqrt{\varepsilon}$.

Corollary 4.1. *Assume the set $W \leq 0$ ($W \geq 0$) is a compact set in phase space, and Assumption 1.1 is valid on its boundary $W = 0$. Assume further that $\bar{F} < 0$ ($\bar{F} > 0$) on $W = 0$. Then, the set $W < b\sqrt{\varepsilon}$ ($W > b\sqrt{\varepsilon}$) is mapped into itself for positive time and ε small enough.*

Proof of Lemma 4.1. Assume $\bar{F} > \mu > 0$. Consider Eq. (2.8) on the O(1)-time scale.

Then for $|W| < \delta$ we have

$$\frac{dW}{dt} = \varepsilon\ \hat{F}^1\ (E,\phi) + O(\varepsilon^2) + O(\varepsilon\delta^2)\ . \tag{4.2}$$

Let

$$\tilde{F} := \hat{F}^1(E,\phi) - \bar{F}(E)$$

and

$$\mathcal{H}\cdot\tilde{F} := \int^{\phi} \tilde{F}\ (E,s)\ ds \quad .$$

Substitute

$$\hat{W} = W + \varepsilon\ \mathcal{H}\cdot\tilde{F}/W$$

By assumption $W > b\sqrt{\varepsilon}$ and hence

$$\varepsilon\ \mathcal{H}\cdot\tilde{F}/W \leqslant (\sqrt{\varepsilon}/b)\ \max\ |\mathcal{H}\cdot\tilde{F}|\ ,$$

so that $|W-\hat{W}| = O(\sqrt{\varepsilon}/b)$. Thus by (4.2)

$$\frac{d}{dt}\ \hat{W} - \varepsilon\ \bar{F}(E) = O(\varepsilon^2) + O(\varepsilon\delta) + O(\varepsilon/b^2)\ .$$

Choosing $\delta>0$ sufficiently small and b sufficiently large we obtain

$$O(\varepsilon^2) + O(\varepsilon\delta) + O(\varepsilon/b^2) < \varepsilon\bar{\mu}/4\ .$$

Hence

$$\frac{d}{dt}\ \hat{W} > \varepsilon\ \mu/4$$

as long as $\bar{F}(E) > \mu/2$, i.e., as long as $E_1 < E < E_2$.

Since $\frac{dE}{dt} = O(\varepsilon)$, we can still decrease δ to yield

$$\bar{F}(E(t)) > \mu/2$$

for $0 \leqslant t \leqslant \delta/3\varepsilon\mu$. Thus, we get at $T = \delta/3\varepsilon\mu$

$$W(t) > \delta\ .$$

The case $\bar{F} < 0$, $\delta < 0$ is equivalent. *QED*

With the help of Lemma 4.1 we can now prove Theorem 2.

Proof of Theorem 2. Let (E_1, E_2) be as in Lemma 4.1. For every $E\varepsilon(E_1, E_2)$ we choose an open set $P^E \varepsilon\ \bar{S}^1$ so that for each $\phi\ \varepsilon\ P^E$

$$\hat{F}^1(E,\phi)>\eta>0, \tag{4.3a}$$

$$\int_{\phi}^{\phi+\theta}\hat{F}^1(E,s)ds > 0, \quad 0 \leqslant \theta \leqslant 2\pi(p^2+q^2)^{-1/2} . \tag{4.3b}$$

The assumption $\bar{F} > 0$ guarantees the existence of P^E open for each $E_1 \leqslant E \leqslant E_2$.

Define now the open set D on the resonant manifold $\{W = 0\}$ by

$$D := \{E, \phi : E_1 < E < E_2, \phi \in P^E\} .$$

By Lemma 4.1 it suffices to show that any solution with initial data in D and $W = 0$ will cross the manifold $W = b\sqrt{\varepsilon}$ at a finite time, where b is an ε-independent constant. Using (3.5) and (4.3), we get, on the $\sqrt{\varepsilon}$-time scale,

$$\dot{\bar{W}} = \hat{F}^1 (E^o, \phi^o) + O(\sqrt{\varepsilon}) > \eta/2$$

for $(E^o.\phi^o) \in D$, provided ε is small enough.

The function

$$Q(\phi) := -\int_{\phi^o}^{\phi} \hat{F}^1 (E^o, \theta)\, d\bar{\theta} \tag{4.4a}$$

is well-defined on the covering $\mathbb{R}$ of $\bar{S}^1$, and negative for $\phi > \phi^o$ by (4.3). The new variable

$$H := 1/2\, \bar{W}^2 + Q(\phi) \tag{4.4b}$$

satisfies

$$\frac{dH}{d\tau} = O(\sqrt{\varepsilon})$$

and hence

$$\left|H(\tau) - H^o\right| = O(\sqrt{\varepsilon}\tau) = O(\varepsilon t) . \tag{4.5}$$

But $\bar{W}$ is a monotonically increasing function in the interval (ϕ^o,ϕ^1), where ϕ^1 is the first zero of $\hat{F}^1$ in the clockwise direction say, and $-Q(\phi)$ admits a positive lower bound on $\phi > \phi^1$. This yields a lower bound on $\bar{W}$,

$$\dot{\bar{W}} > \sqrt{2}\min(-Q(\phi)) = K ,$$

for $\tau > \tau^o$, where τ^o is the time at which $\phi(\tau^o) = \phi^1$. Thus

$$\phi(\tau) > (\tau - \tau^0) K .$$

Let L be an integer for which

$$L > 2b^2 / \bar{F}(E^0)$$

and $\tau > 2\pi L/K + \tau^0$. Assuming ε small enough, so that

$$\tau << 1/\sqrt{\varepsilon} \quad ,$$

we get $\bar{W}(\tau) > b$, or $W(\tau) > b\sqrt{\varepsilon}$. The case $\bar{F}(E^0) < 0$ is completely analogous. *QED*

Corollary 4.2. *If* $\hat{F}^1 (E,\phi) \neq 0$ *for* $E_1 \leqslant E \leqslant E_2$, $\phi \in \bar{S}^1$, *then any point on the resonant manifold restricted to the above energy interval belongs to an escape set.*

In fact, in such a case the set P^E, defined in (4.3), is the whole circle $\bar{S}^1$. Notice that P^E is simply the cross-section of D at E = *const.*

Remark. Theorem 2 does not specify anything about the non-empty escape set D containing a quasi-preserved point or not. Under certain conditions we can make a definite statement on this matter. One such case corresponds to P^E having a hyperbolic point on its boundary. Thus a whole segment of a hyperbolic branch can lie on the boundary of D.

This cannot happen for an elliptic branch for which flow on the ε time scale occurs along the branch (see Theorem 3 below), since slow averaging (Sect. 5) holds up to the degenerate point, $\hat{F}^1 = \partial\hat{F}^1/\partial\phi = 0$, separating it from the adjacent hyperbolic branch (see points 1, 4 and 6 in Fig. 2b). The escape set D always has to contain such a degenerate point, (E_o,ϕ_o) say, on its boundary; then ϕ_o lies in the *interior* of P^{E_o}. In this case, $\hat{F}^2(E_o,\phi_o)$ has the appropriate sign (see points 4 and 6 in the figure), and resonance breaking does occur in finite time for solutions starting *outside* D. The case of approach to D through non-degenerate points along a hyperbolic branch is more delicate and will be handled in Theorem 4.

5. Asymptotic Stability of Resonant Solutions

We now turn to the main object of our paper, namely the investigation of solutions which are trapped for time $O(1/\varepsilon)$ at least in a $O(\sqrt{\varepsilon})$ neighborhood of the resonant manifold. For that purpose, we choose an elliptic branch, cf. Definition 3.1 and Lemma 3.1, $\phi^o(E)$ on an (E_1, E_2) interval, and consider a certain neighborhood of that branch, not necessarily small. Our main observation at this point is that H as given in (4.4) is a slow variable. A somewhat different definition of H will be more useful here.

Definition 5.1. Let $Q(E,\phi)$ be given by

$$\frac{\partial Q}{\partial \phi} = - \hat{F}^1\,(E,\phi) \quad , \qquad Q(E,\ \phi^o(E)) = 0$$

cf. (3.1). Then

$$H^\varepsilon(\bar{W},E,\phi) := \frac{1}{2}\,\bar{W}^2 + Q(E,\phi) + \sqrt{\varepsilon}\,\bar{W}\,\hat{G}^1(E,\phi) \ .$$

Consider now $\bar{W}$ and ϕ formally as a conjugate canonical pair of variables for the Hamiltonian H^ε, where E and ε are constant parameters. The associated equations of evolution are given by

$$\frac{d\phi}{d\tau} = \frac{\partial H^\varepsilon}{\partial \bar{W}} = \bar{W} + \sqrt{\varepsilon}\,\hat{G}^1\,(E,\phi) \quad , \tag{5.1a}$$

$$\frac{d\bar{W}}{d\tau} = -\,\frac{\partial H^\varepsilon}{\partial \phi} = \hat{F}^1(E,\phi) - \sqrt{\varepsilon}\,\bar{W}\,\frac{\partial}{\partial \phi}\,\hat{G}^1 \quad , \tag{5.1b}$$

$$\frac{dE}{d\tau} = 0 \quad . \tag{5.1c}$$

The phase flow of (5.1) is topologically simple if H^ε is a convex function of $\bar{W},\phi$. This is the motivation for the following definition.

Definition 5.2. Let

a) $\bar{H}^{\varepsilon}(E) > 0$, $E_1 < E < E_2$ be given by the condition that

$$V_E^{\varepsilon} := \left\{\bar{W},\ \phi\ :\ H^{\varepsilon}\ (\bar{W},E,\phi) < \bar{H}^{\varepsilon}(E)\right\} \times \left\{E\right\} \tag{5.2a}$$

is a set on which $H^{\varepsilon}(\bar{W},E,\phi)$ is strictly convex in $\bar{W},\phi$ and $(O,E,\phi^{o}(E))\epsilon V_E^{\varepsilon}$. Define

$$V^{\varepsilon} := \bigcup_{E\epsilon(E_1,E_2)} V_E^{\varepsilon} \tag{5.2b}$$

and $\Lambda^{\varepsilon} := V^{\varepsilon} \times \bar{S}^1$, with $\bar{S}^1$ defined as in Lemma 2.1 .

b) The perturbed elliptic branch $(\bar{W}^{p}(E),\phi^{p}(E))$ is given, for each $E\epsilon(E_1,E_2)$, by the unique minimum of $H^{\varepsilon}(\cdot,E,\cdot)$.

Thus, for each $E\ \epsilon(E_1,E_2)$, V_E^{ε} is foliated by the family of closed, convex trajectories of (5.1a, b), while

$$\bar{W}^{\ p}(E) = -\sqrt{\varepsilon}\ \hat{G}^1(E,\phi^{p}) + O(\varepsilon)\ , \tag{5.2c}$$

$$\phi^{p}(E) = \phi^{o}(E) + O(\varepsilon)\ , \tag{5.2d}$$

is the unique critical point of H^{ε} and depends parametrically on E. We proceed by introducing action-angle variables for the Hamiltonian H^{ε}, canonically related to $\bar{W}$, ϕ .

Definition 5.3. The *action* I^{ε} of the Hamiltonian H^{ε} is defined, in the usual way, as

$$I^{\varepsilon}(H;E) = \frac{1}{2\pi} \oint \bar{W}\ d\phi\ ,$$

where the integral is taken over the contour $H^{\varepsilon}(\cdot,E,\cdot) = H$. The conjugate angle variable, ξ^ε, is defined by an appropriate generating function (see Goldstein, 1980; Lichtenberg and Lieberman, 1983).

Using this definition, the set Λ^ε can now be presented as

$$\Lambda^\varepsilon = \left\{ I,E,\xi,\psi : 0 \leqslant I^\varepsilon < \bar{I}^\varepsilon(E),\ E \,\varepsilon (E_1,E_2), \xi\varepsilon S^1, \psi\varepsilon\bar{S}^1 \right\} , \tag{5.3}$$

where $\bar{I}^\varepsilon = I^\varepsilon(\bar{H}^\varepsilon(E),E)$, with $\bar{H}^\varepsilon(E)$ is in (5.2a). The proposition below follows from the strict convexity of H^ε on Λ^ε, by substituting H^ε into $I^\varepsilon(H,E)$.

Lemma 5.1. *If $H^\varepsilon \ \varepsilon\ C^k$, $k > 1$, on Λ^ε as a function of $\bar{W}$, E and ϕ, then $I^\varepsilon(H,E)\ \varepsilon\ C^k(\Lambda^\varepsilon)$.*

Thus

$$\tilde{W}\,(H,E) := \left(\frac{\partial I^\varepsilon}{\partial H} \bigg|_{E=const.} \right)^{-1} \varepsilon\ C^{\,k-1}(\Lambda^\varepsilon) \tag{5.4}$$

is the oscillation frequency of system (5.1) on the contour $H^\varepsilon(\cdot,E,\cdot) = H$. Furthermore, under the same hypothesis as the Lemma above, one can also show the following:

Lemma 5.2 . *The angle variable ξ^ε has k-1 derivatives with respect to $\bar{W}$, E and ϕ on Λ^ε provided $(\bar{W},\phi) \neq (\bar{W}^p,\ \phi^p)$.*

Proof. We can solve for

$$\bar{W} = \bar{W}(I,E,\phi) \quad , \quad I > 0,$$

from

$$I^\varepsilon(\bar{W},\phi,E) = I \quad , \tag{5.5}$$

since

$$\frac{\partial I^\varepsilon}{\partial \bar{W}} = \left.\frac{\partial I^\varepsilon}{\partial H}\right|_E \cdot \left.\frac{\partial H^\varepsilon}{\partial \bar{W}}\right|_{E,\phi} = \frac{1}{\tilde{W}} \left.\frac{\partial H^\varepsilon}{\partial \bar{W}}\right|_{E,\phi} \neq 0$$

if $(\bar{W},\phi) \neq (\bar{W}^p, \phi^p)$. Then $\xi^\varepsilon(I,E,\phi)$ is given by the generating function

$$\xi^\varepsilon = \frac{\partial}{\partial I} S(I,E,\phi) \quad , \tag{5.6a}$$

$$S := \oint^{\phi} \bar{W}\ (I,E,s)\ ds \quad . \tag{5.6b}$$

The fact that $S \in C^k$, and hence $\xi^\varepsilon(\bar{W},E,\phi) \in C^{k-1}$, as long as $(\bar{W},\ \phi) \neq (\bar{W}^p,\phi^p)$, follows by substituting (5.5) into (5.6). *QED*

From the corollary above, and the fact that the transformation of variables $(\bar{W},\phi) \to (I^\varepsilon,\xi^\varepsilon)$ is nondegenerate for $I^\varepsilon > 0$, we conclude that $\bar{W}$ and ϕ have both k-1 continuous derivatives with respect to I^ε, ξ^ε, for $I^\varepsilon \neq 0$. The ε-dependence of $I^\varepsilon,\xi^\varepsilon$ will be suppressed hereafter, unless needed explicitly. We now define slow averaging with respect to the $\sqrt{\varepsilon}$-time scale.

Definition 5.4. Let h be an arbitrary function defined on V_E^ε . The slow average $h^* = h^*(H,E)$ of h is given by

$$h^* := \frac{1}{T} \int_0^T h\ (\bar{W}(\tau),E,\phi(\tau))d\tau \quad , \tag{5.7}$$

where $(\bar{W}(\tau),\phi(\tau))$ is the solution of (5.1) on the level curve $H^\varepsilon = H$ at E fixed, and $T = T\ (H,E)$ is the period of revolution on this level.

We shall use $(\ \cdot\ )^*$ to indicate slow averaging of $(\ \cdot\ )$, by analogy with the fast average (2.6). In terms of the action-angle pair $(I^\varepsilon,\ \xi^\varepsilon)$, slow averaging takes a particularly simple form,

$$h^*(I,E) = \frac{1}{2\pi} \int_0^{2\pi} h(I,E,\xi)\, d\xi \quad . \tag{5.8}$$

Remark. We denote by h^* the average of h in both the $\{H,E\}$, and $\{I,E\}$, representations, Eqs. (5.7) and (5.8) respectively, and specify the functional dependence only when needed. Notice that $H = H^\varepsilon(I,E)$, so that $\left(\frac{\partial h^*}{\partial E}\right)\Big|_H \neq \left(\frac{\partial h^*}{\partial E}\right)\Big|_I$, in general.

For reasons which will become apparent later on, it is useful to define

$$j^\varepsilon := \sqrt{2I^\varepsilon}$$

and we shall drop the ε parameter, unless needed; j is essentially the amplitude of the slow oscillation (see (5.18)). The following lemma is the cornerstone to our main results

Lemma 5.3. *Assume $H^0 \in C^{k+1}$, $\{f_i, g_i\} \in C^k$ in Eq. (1.1). Let $\bar{\tau} := \sqrt{\varepsilon}\tau = \varepsilon t$ be the slow time variable, and $\Lambda_-^\varepsilon = \Lambda^\varepsilon - \{I,E,\xi,\psi : \bar{W} = \bar{W}^P(E),\ \phi = \phi^P(E)\}$. Then, on Λ_-^ε, Eqs. (2.8) take the form:*

$$\frac{dj}{d\bar{\tau}} = A^\varepsilon(j,E,\xi) + \sqrt{\varepsilon}\, h_1^\varepsilon(j,E,\xi,\psi;\varepsilon) \quad , \tag{5.9a}$$

$$\frac{dE}{d\bar{\tau}} = B^\varepsilon(j,E,\xi) + \sqrt{\varepsilon}\, h_2^\varepsilon(j,E,\xi,\psi;\varepsilon) \quad , \tag{5.9b}$$

$$\frac{d\xi}{d\bar{\tau}} = \frac{1}{\sqrt{\varepsilon}} \tilde{W}^\varepsilon(j,E) + h_3^\varepsilon(j,E,\xi,\psi;\varepsilon) \quad , \tag{5.9c}$$

$$\frac{d\psi}{d\bar{\tau}} = \frac{1}{\varepsilon} W^1(E) + \frac{1}{\sqrt{\varepsilon}} h_4^\varepsilon(j,E,\xi,\psi;\varepsilon) \quad ; \tag{5.9d}$$

and a) $W^1 \in C^k$ and $\tilde{W} \in C^{k-1}$, as given by (2.5e) and (5.4), respectively;

b) A^ε and B^ε are C^{k-1} in their variables on Λ^ε_-, with C^{k-2} extensions to Λ^ε, given by

$$A^\varepsilon(j,E,\xi) = \frac{1}{\tilde{W}j}\left\{\bar{W}(\bar{W}-\bar{W}^P)(\hat{F}^1_W + \hat{G}^1_\phi) + \left[\frac{\partial H}{\partial E} - \left(\frac{\partial H}{\partial E}\right)^*\right] \hat{F}^2\right\}, \tag{5.10a}$$

$$B^\varepsilon(j,E,\xi) = \hat{F}^2 , \tag{5.10b}$$

where $\hat{F}^1$, $\hat{G}^2$, $\hat{F}^2$ are all given as in (2.9), expressed in j,E,ξ dependence, all partial derivatives being taken with respect to the $(\bar{W},E,\phi,\psi)$ variables;

c) h_i, $i = 1,\dots,4$ all admit $k-2$ continuous derivatives in Λ^ε_-, h_2 and h_4 can be extended as C^{k-2} functions over Λ^ε, h_1 as C^{k-3} and

$$\lim_{\varepsilon\to 0}\lim_{j\to 0} jh_3 = \bar{h}_3(E,\xi) = c(E)\cos\xi , \tag{5.11}$$

where $c = c(E)$ is a given, smooth function of E, and exists uniformly in E and ξ independently of the order of the limits.

Remark. In all of the above, the dependence on ξ and ψ is periodic, with periods 2π and $2\pi(p^2+q^2)^{1/2}$, respectively.

Proof of Lemma 5.3. Let $j(H,E) > 0$. By the hypotheses, $H^\varepsilon \in C^k$ so that $j \in C^k$ in terms of H and E. Taking the τ derivative yields

$$\frac{dj}{d\tau} = \frac{1}{j}\frac{dI^\varepsilon}{d\tau} = \frac{1}{j}\left(\left.\frac{\partial I^\varepsilon}{\partial H}\right|_E \frac{dH}{d\tau} + \left.\frac{\partial I^\varepsilon}{\partial E}\right|_H \frac{\partial E}{d\tau}\right) . \tag{5.12}$$

By (5.4), $\left.\left(\frac{\partial I^\varepsilon}{\partial H}\right)\right|_E = 1/\tilde{W} \in C^{k-1}$, while one can show, using (5.4) and (5.7), that

$$\left.\frac{\partial I^{\varepsilon}}{\partial E}\right|_{H} = -\frac{1}{\tilde{W}}\left(\left.\frac{\partial H}{\partial E}\right|_{\bar{W},\phi}\right)^{*} = -\frac{1}{\tilde{W}}\left(\left.\frac{\partial Q}{\partial E}\right|_{\bar{W},\phi}\right)^{*} + O(\sqrt{\varepsilon}) \ . \qquad (5.13)$$

The second equality above follows from Definition 5.1 .

Using (5.11) and (2.8) we can compute

$$\frac{dH}{d\tau} = \sqrt{\varepsilon}\ \bar{W}(\bar{W}-\bar{W}^{p})(\hat{F}^{1}_{\bar{W}}+\hat{G}^{1}_{\phi}) + \frac{\partial H}{\partial E}\frac{dE}{d\tau} + \varepsilon h\ (\bar{W},E,\phi;\varepsilon) \quad , \qquad (5.14)$$

where h is C^{k-2} in all its variables, and

$$\partial h/\partial\phi = \partial h/\partial\bar{W} = 0 \text{ at } (\bar{W}^{p},\phi^{p}) \ . \qquad (5.15)$$

Substituting (5.13) and (5.14) into (5.12) and dividing by $\sqrt{\varepsilon}$, we obtain (5.9a) with

$$h_{1} = \frac{1}{j}\left[h + \frac{\bar{W}}{\tilde{W}}\left\{\left.\frac{\partial}{\partial E}\hat{G}^{1}\right|_{\bar{W},\phi} - \left(\left.\frac{\partial}{\partial E}\hat{G}^{1}\right|_{\bar{W},\phi}\right)^{*}\right\}\right] \ . \qquad (5.16)$$

The equations for E and ψ remain the same as in (2.9), up to the new change of variables, while (5.9c) follows by transforming the canonical system (5.1) for H^{ε} to the action-angle variables I^{ε} and ξ^{ε}, and using again (5.4) and (5.6):

$$\frac{d\xi^{\varepsilon}}{d\tau} = \frac{1}{\sqrt{\varepsilon}}\tilde{W}\ (j,E) + h_{3}^{\varepsilon} \ . \qquad (5.17)$$

The differentiability of system (5.9) for $j > 0$ follows from Lemmas 2.2, 5.1 and 5.2. The rest of the statements of Lemma 5.3 will follow from Lemma 5.4, which has an importance of its own.

Lemma 5.4. *Assume $H^{\varepsilon}(\bar{W},\phi,E) \in C^{k}$. Then $\bar{W},\phi$ are C^{k-1} functions of j^{ε}, ξ^{ε} in Λ^{ε}. Moreover, near $(\bar{W},\phi) = (\bar{W}^{P},\phi^{P})$ we have the expansion*

$$\bar{W} = \bar{W}^{P} + (2F^{1/2})^{1/2} j \cos(\xi+\kappa) + \sum_{i=2}^{k-2} j^{i} P_{i}^{\bar{W}}(\xi) + j^{k-1} G^{\bar{W}}(j,\xi) , \qquad (5.18a)$$

$$\phi = \phi^{P} + (2F^{1/2})^{-1/2} j \sin(\xi+\kappa) + \sum_{i=2}^{k-2} j^{i} P_{i}^{\phi}(\xi) + j^{k-1} G^{\phi}(j,\xi) . \qquad (5.18b)$$

Here $\kappa = \kappa(E)$ is an $O(\sqrt{\varepsilon})$ phase shift, $F = F(E) = \frac{\partial}{\partial\phi} \hat{F}(E,\phi(E)) + O(\sqrt{\varepsilon})$, $\left\{P_{i}^{\bar{W}},P_{i}^{\phi}\right\}$ are E-dependent trigonometric polynomials in ξ; $G^{\bar{W}}$, $G^{\phi} \varepsilon C^{k-1}$ and all; the above are C^{k} functions of the parameter E.

The proof of Lemma 5.4 is given in Appendix A. An application of this lemma, together with Lemma 2.2, give the C^{k-1} extensions of B, h_2, and h_4 to Λ^{ε}. The C^{k-2} extension of A to the elliptic branch itself follows from the representation (5.10a) of A. In fact, since $\bar{W}-\bar{W}^{P} = O(j)$, the first term on the right-hand side of (5.10a) is readily seen to have a C^{k-2} extension to $j=0$. The second term contains as a factor the deviation of $\partial H/\partial E$ from its slow average,

$$\frac{\partial H}{\partial E} - \left(\frac{\partial H}{\partial E}\right)^{*} \qquad (5.19)$$

which is a C^{k-1} function in j, with a null first j-derivative at 0. Thus, A has a C^{k-2} extension to $j=0$. As for h_1, it also is made up, cf. (5.16), of two terms. The first term, h, has a critical point at $j = 0$, cf. (5.15), and the second term involves also the deviation from its slow average of some C^{k-1} function. Thus, by using Lemma 5.4 again, we get the C^{k-3} extension for h_1. We leave the proof of Eq. (5.11) for Appendix B and merely remark here that $c(E) > 0$ or $c(E) < 0$ if $\phi^{0}(E)$ is increasing or decreasing in time, respectively. This concludes the proof of Lemma 5.3. *QED*

We are now in a position to state the main result.

Theorem 3. *Consider the system*

$$\frac{dj}{d\tau} = A^{*}(j,E) \ , \tag{5.20a}$$

$$\frac{dE}{d\tau} = B^{*}(j,E) \ , \tag{5.20b}$$

where A^{*}, B^{*} are the averaged functions A^{ε}, B^{ε} given by Lemma 5.3 evaluated at $\varepsilon = 0$; the domain of A^{*}, B^{*} is $\Lambda^{\varepsilon}\_/S^{1}x\bar{S}^{1}$. Assume $\left\{f_{i}, g_{i}\right\}$, $i = 1,2$, in (1.1) are C^{k} functions, $H^{o} \in C^{k+1}$ with $k \geq 4$. Then, for ε sufficiently small:

a) Every hyperbolic critical point P (limit cycle γ) of (5.20) in the domain of A^{*}, B^{*} above corresponds to a two-dimensional (three-dimensional) invariant hyperbolic torus of (1.1), given in terms of E, j, ξ, ψ by an $o(1)$ perturbation of $PxS^{1}x\bar{S}^{1}$ or $\gamma xS^{1}x\bar{S}^{1}$, respectively.

b) Consider

$$D(E) := \hat{F}^{1}_{W} + \hat{G}^{1}_{\phi} + \frac{1}{4|\hat{F}^{1}_{\phi}|} \hat{F}^{1}_{E} \hat{F}^{2}_{\phi}$$

all evaluated at $\phi = \phi^{P}(E)$ (or, equivalently, at $j = 0$). Then, if $D(E) < 0$ on a subinterval $(E_{1}', E_{2}') \subset (E_{1}, E_{2})$, there exists $\delta > 0$, independent of ε, such that any initial data on the cylinder $\mathscr{C}_{\delta}$,

$$\mathscr{C}_{\delta} := \left\{ j, E\colon\ 0 \leq j \leq \delta,\ \ E_{1}' \leq E \leq E_{2}' \right\} \times S^{1}x\bar{S}^{1} \ , \tag{5.21}$$

can leave $\mathscr{C}_{\delta}$ only at $E = E_{1}'$ or $E = E_{2}'$. Furthermore, if

$$B^{*}(E_{2}', 0) < 0 < B^{*}(E_{1}', 0) \ , \tag{5.22}$$

then $\mathscr{C}_{\delta}$ is a subset of an attractor basin of (1.1).

c) Assume

$$B^*(E^o,0) = 0 \quad , \quad \frac{\partial}{\partial E} B^*(E,0)\Big|_{E=E^o} < 0 \quad , \quad E^o \epsilon (E_1,E_2) \ . \qquad (5.23a.b)$$

If the assumptions of (b) also hold, then there exists a δ-neighborhood of $\left\{j = 0,\ E^o\right\} \times S^1 \times \bar{S}^1$ *which attracts solutions of* (1.1)*(see point 2 in Fig. 2b). In general, conditions* (5.23) *guarantee existence of a single periodic orbit in a neighborhood*

$$|\phi-\phi^p(E)| \leq O(\epsilon), \ |W| \leq O(\epsilon), \quad |E-E^o| \leq O(\sqrt{\epsilon})$$

and its period is close to that of the unperturbed system at $E = E^o$. *This periodic solution may be surrounded by a two-dimensional attracting torus at an* $O(\sqrt{\epsilon})$ *distance from the elliptic branch at* $E = E^o$.

A slight generalization on the hypotheses of (b) is given in

Corollary 5.1. *Let δ>0 such that*

$$A^*(E,\delta) < 0,$$
$$B^*(E_2',j) < 0 < B^*(E_1',\ j) \ ,$$

in $0 \leq j \leq \delta$, $E_1' \leq E \leq E_2'$. *Then the cylinder* $\mathscr{C}_\delta$, *with the above δ, is an attractor basin of the flow (not necessarily maximal). In particular, any initial data inside the cylinder give rise to a trapped solution satisfying the resonance relation p/q .*

Proof of Theorem 3 and Corollary 5.1. Consider Eq. (5.9c). If ε is small enough, (5.11) guarantees that, for

$$j > b\sqrt{\epsilon} \quad , \quad b > 0 \ ,$$

we get

$$\left|h_3^\epsilon\right| < \frac{1}{\sqrt{\epsilon}} \frac{c(E)}{b} \ .$$

By choosing

$$b > 2 \max_{E_1' \leq E \leq E_2'} \left|\frac{c(E)}{\tilde{W}(0,E)}\right|$$

(5.9c) yields

$$\frac{d\xi}{d\bar{\tau}} \geqslant \frac{1}{\sqrt{\varepsilon}}\,\frac{\tilde{W}}{2} \gg 1 \;.$$

So, for $j \geqslant b\,\sqrt{\varepsilon}$, we may apply averaging to (5.9a,b) by an $O(\sqrt{\varepsilon})$ change of variables $\tilde{J} = j + O(\sqrt{\varepsilon})$, $\tilde{E} = E - E_0 + O(\sqrt{\varepsilon})$, to obtain

$$\frac{d\tilde{J}}{d\bar{\tau}} = A^*(J,E) + O(\sqrt{\varepsilon}) \quad , \tag{5.24a}$$

$$\frac{d\tilde{E}}{d\bar{\tau}} = B^*(J,E) + O(\sqrt{\varepsilon}) \quad , \tag{5.24b}$$

together with (5.9c,d) appropriately transformed. Notice now that substituting $\varepsilon=0$ into (5.9a,b) changes A, B only by $O(\sqrt{\varepsilon})$ terms so that A^*, B^* can be computed at $\varepsilon=0$ without changing the leading terms in (5.24). The conclusion of (a) follows by linearizing (5.24) near its critical point (limit cycle) and applying the stable/unstable manifold theorem for a system of the above type (see Kelley, 1967, Theorem 4).

The averging process cannot be extended too close to the elliptic branch $j=0$, but A^* and B^* are C^{k-2} in the neighborhood of $j=0$ by Lemma 5.3. Thus

$$A^*(j,E) = \frac{\partial}{\partial j} A^*\Big|_{j=0} \cdot j + O(j^2) \;,$$

since $A^*(0,E) = 0$. Therefore, given $\delta > 0$ small enough, independently of ε, $A^*(j,E) < 0$ for $j \leqslant \delta$, $E \varepsilon (E_1', E_2')$, provided

$$\frac{\partial}{\partial j} A^*\Big|_{j=0} < 0 \quad , \quad E \varepsilon (E_1' \;,\; E_2') \;.$$

The above condition can be shown to be equivalent to $D(E) < 0$ as given in (b), using (5.10) and (5.18). Thus, the first part of (b) follows by considering the averaged system (5.24) at $j=\delta$, away from the elliptic branch, and using the fact that $j=\delta$ separates the four-dimensional phase space.

Turning to the second statement of (b), we consider the averaged system on the set

$$\left\{j, E \; : \; b\sqrt{\varepsilon} < j < \delta, \; E = E_i^1 \; , \quad i=1,2\right\}$$

to conclude that solutions cannot escape $\mathscr{C}_\delta$ through the above set. For $0 < j < b\sqrt{\varepsilon}$ we notice

$$B^\varepsilon(j,E,\xi) = B^\varepsilon(0,E,\xi) + 0(\sqrt{\varepsilon}) = B^*(0,E) + 0(\sqrt{\varepsilon})$$

due to the C^{k-1} differentiability of B at $j=0$. Thus, the *non*-averaged system (5.9a,b) gives trapping within the interval (E_1',E_2'). This concludes the proof of (b) and the corollary.

As for (c), $B^\varepsilon(0,E,\xi) = B^*(0,E) = \hat{F}^2$ on the elliptic branch. Hence we get from (5.23a), by using (5.11) and (B.5), that $c(E^o) = 0$.

Thus, restricting ourselves to the domain

$$\left\{|E-E^o| < a\sqrt{\varepsilon} \quad , \quad j > b\sqrt{\varepsilon}\right\} , \tag{5.25}$$

we obtain

$$|h^3| < \frac{\sqrt{\varepsilon}a|c'(E^o)| + 0(\varepsilon)}{b\varepsilon} = \frac{1}{\sqrt{\varepsilon}} \; \frac{a}{b} \; c'(E^o) + 0(1) \quad .$$

If we choose

$$\frac{a}{b} < \frac{\tilde{W}(0,E)}{|c'(E^o)|}$$

and a,b arbitrary if $c'(E^o) = 0$, we get

$$\frac{d\xi}{d\bar{\tau}} = \frac{\tilde{W}}{\sqrt{\varepsilon}} + h_3 > \frac{1}{\sqrt{\varepsilon}} \left(\tilde{W} - \frac{a}{b}|c'(E^o)|\right) = 0\left(\frac{1}{\sqrt{\varepsilon}}\right) \quad ,$$

uniformly in the domain (5.25). Thus, we can perform the averaging throughout the above domain. Now, we can assume

$$h_1(j,E,\xi,\psi;\varepsilon) = h_1^o\,(j,E,\xi) + O(\varepsilon^{1/2})$$

by carrying the *fast* averaging of Sect. 2 on to the next order. Then we apply the slow averaging to $A^\varepsilon + \sqrt{\varepsilon}h_1^o$ to conclude, with $\tilde{J}$ and $\tilde{E}$ as in (5.24) that

$$\frac{d}{d\tau}\tilde{J} = A^*\,(\tilde{J},\tilde{E}) + \sqrt{\varepsilon}\,(h_1^o)^*(\tilde{J},\tilde{E}) + O(\varepsilon) \quad , \tag{5.26a}$$

$$\frac{d}{d\tau}\tilde{E} = B^*\,(\tilde{J},\tilde{E}) + O(\sqrt{\varepsilon}) \quad , \tag{5.26b}$$

with $h_1^o\ \varepsilon C^{k-3}$ by the assumption of the theorem. Thus, if $h_1^o(0,\ E^o) > 0$ we get an attracting point of (5.26) at

$$\tilde{J} = \sqrt{\varepsilon}\,\frac{h_1^o(0,E^o)}{A^*_{\tilde{J}}(0,E^o)} + O(\varepsilon) \quad .$$

which, for ε small enough, is well imbedded in the domain (5.25). Applying again the stable/unstable manifold theorem to (5.26) together with (5.9c,d) we obtain the existence of an attracting torus in the above neighborhood.

To prove the existence of a unique periodic solution, rather than a torus, with period O(1) satisfying the estimate of Theorem 3, we have to return to the nonaveraged Eqs. (2.4). Then the claim follows by applying the implicit function theorem to the transformed Eqs. (2.4). The full proof is given in Appendix C. *QED*

The existence proof of surviving periodic solution, given in Appendix C, also yields the uniqueness of such a solution in a $O(\sqrt{\varepsilon})$ neighborhood of the stationary point (5.23) on the elliptic branch. We cannot, however, conclude anything about long-period solutions in a $O(\varepsilon)$ neighborhood of such point, nor the details of the flow there. This is due to the fact that slow averaging cannot be carried out inside the domain (5.25), unless the exceptional case $c'(E^0) = 0$ holds. Outside a certain neighborhood of the stationary points, we conclude that the flow in the annulus

$$N := \left\{ b\sqrt{\varepsilon} < j < \delta \quad , \quad |B^*(0,E)| > \delta \right\}$$

is transversal to the energy surface, provided b is large enough and δ is small enough, both independently of ε.

The flow in N also oscillates around the elliptic branch with an $O(\sqrt{\varepsilon})$ period due to the natural oscillation of the Hamiltonian system (5.1). Inside a $O(\sqrt{\varepsilon})$ neighborhood of the elliptic branch, the above oscillations stop and reverse direction at the contour:

$$0 \leqslant j = \frac{-\sqrt{\varepsilon}\; c(E)\; \sin \xi}{\tilde{W}\;(E,0)} \quad , \tag{5.27}$$

as can be checked out by setting

$$\frac{d\xi}{d\tau} = 0$$

in (5.9c), and using (5.11).

In Fig. 3 we illustrate the flow inside a $\sqrt{\varepsilon}$ neighborhood of the elliptic branch at nonstationary points, where $\phi^0(E)$ is clockwise increasing in time. In the case $\phi^0(E)$ increases in the opposite direction, the whole phase portrait should be reflected in the vertical axis since $c(E) > 0$ in this case (see remark at the end of the proof of Lemma 5.3).

[Fig. 3 near here, please]

So far we concentrated on the small neighborhood of a hyperbolic branch $\phi = \phi^o(E)$ (Theorem 1) or on the arbitrary neighborhood (not necessarily small) of an elliptic branch (Theorem 3). The method of slow averaging clearly fails very close to the separatrix associated with a given hyperbolic branch (see Figs. 2b and 4), since the period of the oscillations becomes unbounded and the distinction between the $\sqrt{\varepsilon}$ time scale of the oscillations and the ε time scale of energy evolution breaks down. In particular the two equations (5.20) no longer represent system (2.4) to leading order very close to the separatrix: there the evolution of solutions depends essentially on the phase ξ as well as on the action variables E, j. To analyze the flow near the separatrix it helps to define the E-dependent functional

$$M(E) := \int_{-\infty}^{\infty} \bar{W}^{2}(\hat{F}^1_W + \hat{G}^1_\phi + \frac{1}{2}\hat{F}^2_E)\, d\tau - \frac{1}{2}\frac{\partial}{\partial E}\int_{-\infty}^{\infty} \bar{W}^2 \hat{F}^2\, d\tau \quad , \qquad (5.28a)$$

where the integrand is taken over the separatrix of the Hamiltonian $H \equiv H^{\varepsilon}(\varepsilon=0)$ in (5.1), evaluated for E fixed.

Remark. This definition is inspired by the well known Melnikov (1963) function, hence the use of M in the notation.

[Fig. 4 near here, please]

Theorem 4. a) *$M(E)$ is well defined for every value of E inside the domain of a hyperbolic branch $\phi = \phi^h(E)$ (Definition 3.1).*

b) *Assume*

$$M(E) < 0 \ , \quad E_1 < E < E_2 \ , \qquad (5.28b)$$

where (E_1, E_2) *is an interval on which both an elliptic branch* $\phi^e(E)$ *and an adjacent hyperbolic branch* $\phi^h(E)$ *are well defined* (Fig. 5). *Then there exists* $\delta(\varepsilon)$, *with* $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, *such that any solution of* (1.1), *initially in the domain:*

$$D := \left\{\bar{W}, E, \phi, \psi:\ \bar{H}(E) - H(\bar{W}, E, \phi) > \delta(\varepsilon),\ E_1 < E < E_2\right\}$$

can escape resonance (*i.e., cross the separatrix of H*) *only at some* $E > E_2$ *or* $E < E_1$.

In particular, condition (5.28) guarantees trapping of *all* solutions inside the separatrix of H, up to a set of arbitrarily small measure for ε small enough.

Proof: The convergence of $M(E)$ follows from the nondegeneracy condition of the hyperbolic branch (3.1b). Consider now a solution of (5.1) with Hamiltonian

$$H(\bar{W}, E, \phi) = \bar{H}(E) - \delta \qquad E \in (E_1, E_2)\ , \tag{5.29}$$

where $\bar{H}(E) := Q(E, \phi^h(E))$, and let $T(\delta, E)$ be the oscillation period of this solution for fixed E in the given interval. Assertion (b) of the theorem will follow if we show that, at time $T = T(\delta, E)$,

$$\bar{H}(E') - H(\bar{W}', E', \phi) > \delta\ , \tag{5.30}$$

where W', E', ϕ', are given as the solution of (2.8) at time $T(\delta, E)$.

The left-hand side of (5.30) is estimated by

$$\int_0^{T(\delta,E)} \frac{d}{d\tau}\left[\bar{H}(E) - H(\bar{W},E,\phi)\right] d\tau + 0(\varepsilon e^{T(\delta,E)}) \ , \tag{5.31}$$

where the integrand is taken again over the contour (5.29), for the same, fixed E. Since the integrand of (5.31) is periodic with period $T(\delta,E)$, the integration limits can be changed to $\left\{-T/2,\ T/2\right\}$. Using (5.9b), (5.10b) and (5.14), one can write (5.31) as

$$\sqrt{\varepsilon} \int_{-T/2}^{T/2} \left\{\frac{\partial}{\partial E}\ (\bar{H}(E) - H(\bar{W},E,\phi))\ \hat{F}^2 - \bar{W}^2\ (\hat{F}^1_W + \hat{G}^1_\phi)\right\} d\tau + 0(\varepsilon T) + 0(\varepsilon e^T). \tag{5.32}$$

Using integration by parts, the T-periodicity of the integrand and the identity $\bar{H}(E) - H = \bar{W}^2/2$ (cf. Definition 5.1), we take the derivative $\partial/\partial E$ outside the integral sign in (5.32) and rewrite it as

$$-\sqrt{\varepsilon} \int_{-T/2}^{T/2} \bar{W}^2 \left(\hat{F}^1_W + \hat{G}^1_\phi + \frac{1}{2}\hat{F}^2_E\right) + \frac{1}{2}\ \frac{\partial}{\partial E} \int_{-T/2}^{T/2} \bar{W}^1\ \hat{F}^2\ d\tau\ .$$

As $\delta \to 0$, $T(\delta,E) \to \infty$ and the integral above converges to $-\sqrt{\varepsilon}\ M\ (E)$. The result of the theorem follows by choosing ε, δ sufficently small so that

$$-\sqrt{\varepsilon}\ M(E) > 0(\varepsilon e^{T(\delta,E)}) + 0(\varepsilon T(\delta,E)) + 0(\sqrt{\varepsilon}\delta)\ . \qquad QED$$

Remark. If system (1.1) is of *Hamiltonian* type, and depends explicitly on time due to a slow parameter $\lambda(\varepsilon t)$, system (5.20) is degenerate and Theorem 3 is void. In fact. we get in this case

$$\hat{F}^2 = (\frac{\partial}{\partial\lambda} H^0) \frac{d\lambda}{d\bar{\tau}} = B^*(E) \ ,$$

while the second term in the definition (5.10a) of A^ε is identically zero. As for the first term of A^ε, it can be shown that:

$$\bar{W}2(\hat{F}^1_W + \hat{G}^1_\phi) = \bar{W}2 \ f(E) \ \frac{\partial}{\partial\phi} \langle H\rangle \quad ,$$

where $\langle H\rangle$ is the ψ average of the Hamiltonian perturbation inducing $\{f_i, g_i\}$ in (1.1) and $f(E)$ a given function. The term above vanishes under slow averaging with respect to ξ, and thus $A^* = O(\sqrt{\varepsilon})$. Thus, (5.26a) indicates that j, or I, is an adiabatic invariant and undergoes $O(\sqrt{\varepsilon})$ variation over an $O(\varepsilon^{-1})$ time interval.

For general perturbations $\{f_i, g_i\}$ however system (1.1) is often *dissipative*. A condition for dissipative behavior over $O(\varepsilon^{-1})$ time can be stated, in terms of the twice-averaged system (5.20), as

$$\frac{\partial A^*}{\partial j} + \frac{\partial B^*}{\partial E} < 0 \quad . \qquad (5.33)$$

Near a stable elliptic point this condition is certainly satisfied, since both $\partial A^*/\partial j$ and $\partial/B^*/\partial j$ are negative. In a neighborhood of a more general attractor, cf. Theorem 3, condition (5.33) does not have to hold at every point of the attractor basin, but does hold asymptotically as orbits approach the attractor.

6. Summary and Discussion

We have studied the neighborhood of a given resonant manifold $M_{p,q}$ of an integrable Hamiltonian, in the presence of general perturbations (1.1,1.2). Under suitable technical conditions, it was shown that the Poincaré-Birkhoff theorem for Hamiltonian perturbations in two degrees of freedom generalizes as follows: 1) There exist pairs of alternating elliptic and hyperbolic quasi-preserved periodic solutions, appearing as points on a suitably reduced, two-dimensional representation of $M_{p,q}$ (Lemma 3.1). 2) Near hyperbolic points there exist open escape sets of initial data on $M_{p,q}$ which lead to solutions leaving in finite time an ε-independent δ-neighborhood of $M_{p,q}$ (Theorem 2). 3) Near elliptic points, invariant manifolds of solutions exist and are stable in the large, having one, two or three dimensions (Theorem 3 and Corollary 5.1).

An attempt to draw a global phase portrait of solutions from these results has to take into account the whole set of resonant manifolds in phase space. In fact, since resonant tori constitute a dense set, there exists a resonant manifold $M_{p,q}$ arbitrarily close to any point in phase space.

However, a careful consideration of the averaging procedure of Sect. 2 will indicate that any integration over the fast phase ψ will contribute an $O(\sqrt{p^2+q^2})$ term. Since in the process of averaging we had to integate twice with respect to the fast phase (see definition of u_1 in Eq. (2.7a)) the $O(\varepsilon)$ terms on the right-hand side of (2.8) are, in fact, of order $\varepsilon(p^2+q^2)$. For our analysis to remain valid, these terms should be kept small with respect to the $O(\sqrt{\varepsilon})$ terms. Thus we may define a near-resonance relation by

$$\sqrt{\varepsilon}(p^2+q^2) << 1$$

for any p,q mutually prime. Hence, for given ϵ, we need consider only a finite set of resonant manifolds, and this set will increase as ϵ becomes smaller.

It appears therefore at first that the smaller the perturbation, the larger the possibility of trapping by high-order resonances. This is, however, not the case. For high-order resonance, the deviation of $\hat{F}^1(E,\phi)$ from its torus average $\bar{F}(E)$ (Eq. (4.1a)) becomes small,

$$\hat{F}^1(E,\phi) = \bar{F}(E) + O(\delta) ,$$

where δ is a new ordering parameter which depends on the order p^2+q^2 of the resonance and the smoothness of $\hat{F}^1$. Since $\bar{F}(E)$ has, generically, a discrete set of nondegenerate zeroes on any E-interval, and no degenerate zeroes, $\hat{F}^1$ will have, for high enough resonances, a definite sign on any E-interval, excluding small regions near the roots of $\bar{F}$. By applying Corollary 4.1 we find that any part of the resonant manifold for which $\hat{F}^1(E,\phi) \neq 0$ on the whole torus labeled by E will escape the resonance. Near the zeroes of $\bar{F}$ the function $\hat{F}^2(E,\phi)$, which can also be approximated in this case by a function of the energy alone, is generically nonzero, and the solution will be pushed away from the zero neighborhoods of $\bar{F}$, thus escaping the resonance. Hence, capture by a high-order resonance is really an exceptional phenomenon in the presence of any dissipation.

One may try to explain then, in the spirit of the observation above, the low-order resonance relation 2:5 observed for the periods of revolution of Jupiter and Saturn. The Sun, Jupiter and Saturn constitute the most massive bodies in the solar system, and their motion, neglecting other bodies, is governed by equations with six degrees of freedom in center-of-mass coordinates.

However, two pairs of degrees of freedom are degenerate, due to the existence of additional invariants, besides energy and momentum (Goldstein, 1980), so there are only two independent frequencies, corresponding to the periods of revolution of the planets on their unperturbed ecliptics. Hence, the analysis can, in principle, be carried out in a way similar to the case of two degrees of freedom treated here. In order to find, however, the actual domains of resonance capture, we have to know more about the non-Hamiltonian part of the perturbation.

The idea that resonances among the Galilean satellites of Jupiter are due to tidal dissipation goes back to Laplace. Tidal effects are used routinely in calculating the ephemerides of the moon. Still, much less is known quantitatively about the effects of tides, radiation pressure and other non-conservative phenomena in the solar system (Buys and Ghil, 1984, and further references therein), than about the gravitational interaction between the system's main bodies, on which attention has justifiably focused during the last 200 years of celestial mechanics (Deprit *et al.*, 1984). Motivation to study dissipative and forcing effects comes from the fact that proxy data on periodicities of orbital parameters millions of years ago are starting to appear in the paleoclimatological literature. Hence quantitative verification of actual orbital calculations valid over 10^7-10^8 years might be possible in the near future (*ibid.*).

To consider the full set of 8 resonance relations suspected to exist between the solar system's 9 planets (Molchanov, 1969), one has to extend the analysis to more than two degrees of freedom. This extension is by no means trivial, since slow averaging (Sect. 5) is in general impossible for $n > 3$. Still, certain explicit conditions can be derived under which part of the asymptotic stability results presented here apply. These generalizations will be taken up in a separate paper.

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Appendix A. Proof of Lemma 5.4

By an E-dependent translation and rotation of the coordinates we may assume $(\phi^p, \bar{W}^p) = 0$ and H^ε takes the form:

$$H^\varepsilon = \frac{1}{2}\bar{W}^2 + F_\phi{}^2 + \sum_{3\leqslant i+\ell\leqslant k} a_{i\ell}\,\phi^i\,\bar{W}^\ell + h \quad , \tag{A.1}$$

$h = 0(\phi^k + \bar{W}^k) \in C^0$, and all coefficients have a C^k dependence on E. Introduce now the canonical set of coordinates I^o, ξ^o by the transformation

$$\bar{W} = (2I^o(2F)^{1/2})^{1/2}\cos\xi^o \ , \tag{A.2a}$$

$$\phi = (2I^o(2F)^{-1/2})^{1/2}\sin\xi^o \quad . \tag{A.2b}$$

With (A.2), Eq. (A.1) takes the form:

$$H^\varepsilon = \sigma^o I^o + \sum_{3\leqslant i\leqslant k} I^{o\,1/2} P_i(\xi^o) + I^{o\,k/2}\,G(I^o, \xi^o) \ , \tag{A.3}$$

where P_i are trigonometric polynomials in ξ^o, G is C^1 in $\sqrt{I^o}$ and $\sigma^o = (2F)^{1/2}$.

The proof proceeds in 2 steps:

a) There exists a canonical transformation: $(I^o, \xi^o) \to (\bar{I}, \bar{\xi})$, analytic in $\sqrt{I^o}$, ξ^o, which transforms H^ε into

$$H^\varepsilon = \sigma^{(k)}(\bar{I}) + \bar{I}^{k/2}\,\bar{G}(\bar{I},\bar{\xi}) \quad , \tag{A.4}$$

where

$$\sigma^{(k)} = \sigma^o\,\bar{I} + \sigma^1\,\bar{I}^{3/2} + \ldots + \sigma^{k-2}\,\bar{I}^{k/2}$$

and $\bar{G}$ is continuous. For brevity the E dependence of all the terms has been suppressed.

b) There exists a canonical transformation $(\bar{I}, \bar{\xi}) \to (\bar{\bar{I}}, \bar{\bar{\xi}})$ taking (A.4) into

$$H^{\varepsilon} = \bar{\sigma}\,(\bar{\bar{I}}) \tag{A.5a}$$

with

$$\bar{\bar{I}} = \bar{I} + \bar{I}^{\frac{k}{2}} G^{1} , \tag{A.5b}$$

$$\bar{\bar{\xi}} = \bar{\xi} + \bar{I}^{\frac{k-2}{2}} G^{2} , \tag{A.5c}$$

and G^{1}, G^{2} continuous.

Proof of (a). Assume we have the transformation $(I^{o}, \xi^{o}) \to (I', \xi')$, analytic in $\sqrt{I^{o}}$, by which

$$\bar{W} = (2I'(2F)^{1/2})^{1/2} \cos \xi' + \sum_{i=2}^{k} I'^{\,i/2} P_{i}^{W}(\xi') + I'^{\,k/2} G^{W} , \tag{A.6a}$$

$$\phi = (2I'(2F)^{-1/2})^{1/2} \sin \xi' + \sum_{i=2}^{k} I'^{\,i/2} P_{i}^{\phi}(\xi') + I'^{\,k/2} G^{\phi} , \tag{A.6b}$$

in terms of which H^{ε} takes the form

$$H^{\varepsilon} = \sigma^{(n-1)}(I') + \sum_{n \leq i \leq k} I'^{\,i/2} P_{j}^{(n)}(\xi') + I'^{\,k/2} G^{(n)}(I', \xi') , \tag{A.7}$$

where $n \geq 3$, while P and G are as in (A.3). Introduce now the generating function

$$S = I'' \xi' - \frac{1}{\sigma^{o}} I''^{\,n/2} \mathcal{J} \cdot P_{n}^{(n)}(\xi') ,$$

where $\mathcal{J} \cdot P = (1/2\pi) \int (P - P^{*}) d\xi$, P^{*} being the period average of P. Then, the transformation $(I', \xi') \to (I'', \xi'')$ can be given in an implicit form

$$I' = I'' - \frac{1}{\sigma^0} I''^{n/2} P_n^{(n)}(\xi') \; ; \tag{A.8a}$$

$$\xi'' = \xi' - \frac{n}{2\sigma^0} I''^{(n-2)/2} \mathcal{J}\cdot P_n^{(n)}(\xi') \quad . \tag{A.8b}$$

It is evident that (A.8) is analytic in $\sqrt{I'}$ and ξ' if I' is small enough. The transformation (A.8) can be written explicitly by a convergent power series in $(I'')^{1/2}$. In particular, applying trigonometric polynomials on both sides of (A.8b) and expanding into a Taylor series up to desired order, we may introduce

$$P_n^{(n)}(\xi') = P_n^{(n)}(\xi'') + \sum_{i=n}^{k+2} I''^{(i-2)/2} \tilde{P}_i(\xi'') + I''^{(k+1)/2} \tilde{G}(I'',\xi'') \; ,$$

where $\tilde{G}$ is an analytic remainder and $\tilde{P}_i$ are trigonometric polynomials.

One can easily verify that, by this transformation, (A.6) keeps its general form, with a proper change of the definition of P_j^W, P_j^ϕ , $j \geq n-2$, and G^W, G^ϕ. Substituting (A.8) into (A.7) we obtain

$$H^\varepsilon = \sigma^{(n)}(I'') + \sum_{i \geq n+1}^{k} I''^{i/2} P_i^{(n+1)}(\xi'') + I''^{k/2} G^{(n+1)}(I'', \xi'') \quad ,$$

where

$$\sigma^{(n)}(I) = \sigma^{(n-1)}(I) + (P_n^{(n)})^* I^{n/2}$$

P^* being, as before, the period average of P. The proof of (a) is completed by induction on n up to $n = k$.

Proof of (b). Using (a) we can define $\bar{I}$, $\bar{\xi}$ by which H^ε takes the form (A.4) and (A.6) holds. Define the new action $\bar{\bar{I}}$, as a function of H^ε, by

$$\bar{\bar{I}}\,(C) = \oint \bar{I}\; d\,\bar{\xi} \quad , \tag{A.9}$$

where the integral is taken over the contour $H^{\varepsilon} = C$. On $H^{\varepsilon} = C$, we get, by (A.4),

$$\sigma^{k}(\bar{I}) = C + O(\bar{I}^{k/2})$$

and, since $d\sigma^{(k)}/d\bar{I} \neq 0$ for $\bar{I}$ small enough,

$$\bar{I} = C' + O(\bar{I}^{k/2}) \; ,$$

where $\sigma^{(k)}(C') = C$. Thus, from (A.9),

$$\bar{\bar{I}}(C') = \bar{\bar{I}}(C) + O(\bar{\bar{I}}^{k/2}) \; .$$

The angular variable $\bar{\bar{\xi}}$ can now be calculated by means of the generating function

$$S(\bar{\bar{I}},\bar{\xi}) = \int^{\bar{\xi}} \bar{I}(\bar{\bar{I}},\bar{\xi})d\bar{\xi} = \bar{\bar{I}}\;\bar{\xi} + O(\bar{I}^{k/2}) \; ,$$

from which we get

$$\bar{\bar{\xi}} = \bar{\xi} + O(\bar{I}^{(k-2)/2}) = \bar{\xi} + O(\bar{\bar{I}}^{(k-2)/2}) \; .$$

Substituting $\bar{\bar{I}}$, $\bar{\bar{\bar{\xi}}}$, in (A.6), instead of $I' \equiv \bar{I}$ and $\xi' \equiv \bar{\xi}$, we can verify that the expansion is unchanged up to power k-1 of $\sqrt{\bar{I}}$. The final form (5.18) is obtained by substituting $j = (2\bar{\bar{I}})^{1/2}$ and applying again the translation and small $(O(\sqrt{\varepsilon}))$ rotation to the original variables $\bar{W}$, ϕ.

Appendix B. Behavior of the $O(\sqrt{\varepsilon})$ Frequency near the Elliptic Branch

We prove here Eq. (5.11) in Lemma 5.3. Introduce the new variables:

$$p = F^{-1/4}\sqrt{I}\ \sin\ (\ \xi + \kappa) = (2\ F^{1/2})^{-1/2}\ j\ \sin\ (\xi + \kappa)\ , \qquad \text{(B.1a)}$$

$$q = F^{1/4}\ \sqrt{I}\ \cos\ (\xi + \kappa) = (2\ F^{1/2})^{1/2}\ j\ \cos\ (\xi + \kappa)\ , \qquad \text{(B.1b)}$$

where F, κ are as in (5.18). It is easily shown that (p,q) are a canonical pair for H^{ε}, related by a canonical transformation to $\bar{W},\phi$. By (5.18) we get

$$\bar{W} = \bar{W}^{p} + q + j^2\ G^2\ (j,\ E,\ \xi)\ \ , \qquad \text{(B.2a)}$$

$$\phi = \phi^{p}(E) + p + j^2\ G^1\ (j,\ E,\ \xi)\ , \qquad \text{(B.2b)}$$

and G^1, G^2 ε C^{k-1} in all variables by Lemma 5.4. From (B.1) one then obtains

$$j\ \frac{d\xi}{d\tau} = \left[(2F^{1/2})^{-1/2}\sin\ (\xi + \kappa)\ \frac{dq}{d\tau} - (2F^{1/2})^{1/2}\ \cos\ (\xi + \kappa)\ \frac{dp}{d\tau}\right]$$

$$+ j\left[\frac{1}{4}\left\{\ \sin 2(\xi + \kappa)\right\}\frac{dF}{d\tau} - \frac{d\kappa}{d\tau}\right] \quad . \qquad \text{(B.3)}$$

As $F = F(E)$ and $\kappa = \kappa(E)$ are both smooth by Lemma 5.4, and $dE/d\tau$ tends to a definite limit as $j \to 0$ by the C^{k-1} extension of the right-hand side of (5.9b), the second term of (B.3) tends uniformly to 0 with j. As for the first term, we get by (B.2):

$$\frac{dp}{d\tau} = \frac{d\phi}{d\tau} - \frac{d\phi^{p}}{d\tau} + j\ \frac{dj}{d\tau}\ G^1 + 0(j^2)\ \frac{d\xi}{d\tau}\ \ , \qquad \text{(B.4a)}$$

$$\frac{dq}{d\tau} = \frac{d\bar{W}}{d\tau} - \frac{d\bar{W}^{p}}{d\tau} + j\ \frac{dj}{d\tau}\ G^1 + 0(j^2)\ \frac{d\xi}{d\tau}\ \ . \qquad \text{(B.4b)}$$

Since $d\phi/d\tau$, $d\bar{W}/d\tau$, $dj/d\tau$ all have a bounded limit as $j \to 0$, and $\bar{W}^p, \phi^p$ are smooth in E we conclude, upon substituting (B.4) in (B.3), that $j\ d\xi/d\tau$ has a definite limit as $j \to 0$.

In order to evaluate this limit, one sets $j = 0$ in (B.4) to yield

$$\left.\frac{dp}{d\tau}\right|_{j=0} = \frac{d\phi}{d\tau} - \frac{d\phi^p}{d\tau} \quad , \tag{B.5a}$$

$$\left.\frac{dq}{d\tau}\right|_{j=0} = \frac{d\bar{W}}{d\tau} - \frac{dW^p}{d\tau} \quad . \tag{B.5b}$$

We now evaluate the leading terms, up to $O(\sqrt{\varepsilon})$, of (B.5) which contribute to (B.3) in the limit $j=0$. To evaluate the contribution of $d\phi/d\tau$ and $d\bar{W}/d\tau$ we use (2.9a,b) and compare it to the Hamiltonian system (5.1).

Those terms on the right-hand side of (2.9) which equal the ones present in (5.1) will contribute only to the value of $\tilde{W}(j,E) = d\xi/d\tau$ (compare (5.4) and (5.9c)), which is uniformly bounded at $j=0$. Thus, nonzero contributions to $j\ d\xi/d\tau$ at $j=0$ will come only from those terms in (2.9a,b) which deviate from the Hamiltonian formalism of (5.1). Since $O(1)$ terms are identical in both systems, we consider $O(\sqrt{\varepsilon})$ terms. These terms are also identical in both systems for $d\phi/d\tau$, while comparing (2.9a) to (5.1b) shows that $d\bar{W}/d\tau$ includes a deviation

$$\sqrt{\varepsilon}\ \bar{W}(\hat{G}^1_\phi + \hat{F}^1_W) \ .$$

At $j = 0$, $\bar{W} = \bar{W}^p$ which is of order $\sqrt{\varepsilon}$ (see (5.2c)). Thus, $d\bar{W}/d\tau$ contributes nothing to $O(\sqrt{\varepsilon})$, and the only contribution to this order in (B.3) comes from

$$\frac{d\phi^p}{d\tau} = \frac{d\phi^o}{d\tau} + 0(\varepsilon) = \frac{d\phi^o}{dE}\hat{F}2 + 0(\varepsilon) .$$

Substituting the above in (B.3), dividing by $\sqrt{\varepsilon}$ (upon transforming to the $\bar{\tau} = \sqrt{\varepsilon}\,\tau$ time scale) and letting $j \to 0$ we get

$$j\frac{d\xi}{d\bar{\tau}} = (2\,F^{\frac{1}{2}})^{\frac{1}{2}} \cos\xi \frac{d\phi^o}{dE}\hat{F}2 .$$

By evaluating $d\phi^o/dE$ from definition 3.1 we obtain $c(E)$ of (5.11) as:

$$c(E) = (2F^{\frac{1}{2}})^{\frac{1}{2}}\frac{d\phi^o}{dE}\hat{F}2 = -\sqrt{2}\,F^{-3/4}\frac{d}{dE}\hat{F}1\Big|_{j=0}\hat{F}2 . \qquad \text{(B.5)}$$

Appendix C. The Existence of Unique Periodic Solutions

We introduce the following variables in a $0(\sqrt{\varepsilon})$ neighborhood of an elliptic branch (W^p, ϕ^p):

$$W' = W - W^p(E^o) , \qquad \text{(C.1a)}$$

$$\phi' = \phi - \phi^p(E^o) , \qquad \text{(C.1b)}$$

$$E' = E - E^o , \qquad \text{(C.1c)}$$

and drop the primes for convenience. Substituting these variables into (2.4), we eliminate (2.4d) by using ψ as the independent variable. Denoting ψ by t we are led to the nonautonomous system

$$\dot{W} = \varepsilon U_1(W, E, \phi, t; \varepsilon) , \qquad \text{(C.2a)}$$

$$\dot{E} = \varepsilon U_2(W, E, \phi, t; \varepsilon) , \qquad \text{(C.2b)}$$

$$\dot{\phi} = W + \varepsilon\, V_2(W,\ E,\ \phi,\ t;\ \varepsilon)\ , \tag{C.2c}$$

where U_1, U_2, V_2 are C^k in all variables and with fixed period in t, which we shall take for simplicity to be 2π.

From (C.1) and the definition of E^o, ϕ^p, we get

$$P\cdot U_1 = P\cdot U_2 = 0$$

at $\varepsilon = W = E = \phi = 0$, where P stands for the t-period average. Define now

$$\bar{G}_i\ (z_2,\ z_3) := P\cdot U_i(0,\ z_2,\ z_3,\ t;\ 0),\ i = 1,\ 2.$$

Then, $\bar{G}_i$ for $i = 1, 2$ is k times Lipschitz continuous by assumption, and

$$\bar{G}_1(0,0) = \bar{G}_2\ (0,0) = 0\ . \tag{C.3}$$

The statement of Theorem 3(c) follows from the following.

Lemma. *Consider (C.2) with the assumptions above. If, in addition,*

$$J(\bar{G}) = \frac{\partial}{\partial z_2}\bar{G}_1 \frac{\partial}{\partial z_3}\bar{G}_2 - \frac{\partial}{\partial z_3}\bar{G}_1 \frac{\partial}{\partial z_2}\bar{G}_2 \neq 0 \tag{C.4}$$

holds at $z_2 = z_3 = 0$, then there exists an isolated 2π-periodic solution of (C.2), which is $O(\varepsilon)$ in the maximum norm.

In fact, one can easily check that (C.4) is equivalent to (5.23) by the definition of $B^*\big|_{j=0}$ and that of the elliptic branch (Definition 3.1).

Proof of the Lemma. The proof is a generalization of the method introduced by Hale (1980, Sec. 5.3) for non-hyperbolic systems. Define the Banach space $\mathbb{B}$ as

$$\mathbb{B} := \left\{ z_1(t),\ z_2(t),\ z_3(t)\ :\ z_i(t) \text{ is continuous and } 2\pi\text{-periodic} \right\},$$

with the norm $\|z\| = \max_{i=1,2,3} (\max_t |z_i(t)|)$. Let $\mathcal{K}$ be as in Definition 2.1. For each $\varepsilon > 0$ and constants α_1, α_2, α_3 define the operator $T{:}\mathbb{B}\to\mathbb{B}$ componentwise as

$$T_1 = \varepsilon\alpha_1 + \varepsilon\mathcal{K}(I-P)U_1\ (z_1(t),\ z_2(t),\ z_3(t),\ t;\ \varepsilon)\ , \tag{C.5a}$$

$$T_2 = \alpha_2 + \varepsilon\mathcal{K}(I-P)U_2\ (z_1(t),\ z_2(t),\ z_3(t),\ t;\ \varepsilon)\ , \tag{C.5b}$$

$$\begin{aligned} T_3 = \alpha_3 &+ \varepsilon\mathcal{K}\cdot\mathcal{K}(I-P)U_1(z_1(t),\ z_2(t),\ z_3(t),\ t;\ \varepsilon) \\ &+ \varepsilon\mathcal{K}(I-P)V_2\ (z_1(t),\ z_2(t),\ z_3(t),\ t;\ \varepsilon)\ . \end{aligned} \tag{C.5c}$$

It is evident that T is continuously dependent on ε, α_i. By the Lipschitz property of U_i, V_2 we can find $\varepsilon > 0$ for which T maps the unit ball into itself and is contracting. Hence, we find a fixed point $\vec{z} = \vec{z}\ (\vec{\alpha},\ t,\ \varepsilon)$ of T for any $\vec{\alpha} = (\alpha_1,\ \alpha_2,\ \alpha_3)$ and ε small enough. It can be checked easily that $\vec{z}(\vec{\alpha},t;\varepsilon)$ is continuous in $\vec{\alpha}$, ε and Lipschitz in $\vec{\alpha}$ (by contraction). By the definition of $\mathcal{K}$, $\vec{z}$ is also differentiable in t.

Taking the t derivative of (C.5) we get

$$\dot{z}_1 = \varepsilon(U_1 - P\cdot U_1)\ , \tag{C.6a}$$

$$\dot{z}_2 = \varepsilon(U_2 - P\cdot U_2)\ , \tag{C.6b}$$

$$\dot{z}_3 = \varepsilon\mathcal{K}(I-P)U_1 + \varepsilon(I-P)V_2\ . \tag{C.6c}$$

The last equation can be written, due to (C.5), as

$$\dot{z}_3 = \varepsilon(I-P)V_2 + z_1 - \varepsilon\ \alpha_1\ , \tag{C.7}$$

where we substitute $\vec{z} = \vec{z}(\vec{\alpha},t;\varepsilon)$ in (C.6) and (C.7).

Now $\vec{z}(\vec{\alpha},t;\varepsilon)$ is a solution of (C.2) provided

$$P\cdot U_1 = 0\ ,\quad P\cdot U_2 = 0\ ,\quad P\cdot V_2 + \alpha_1 = 0\ . \tag{C.8}$$

For $\varepsilon = 0$, we get by (C.5):

$$z_1\ (\vec{\alpha},t;0) = 0,\quad z_2(\vec{\alpha},t;\varepsilon) = \alpha_2\ ,\quad z_3(\vec{\alpha},t;0) = \alpha_3\quad .$$

Let $\alpha_2 = \alpha_3 = 0$. Then, by (C.3), at $\varepsilon \overset{-}{=} 0$

$$P\cdot U_1 = P\cdot U_2 = 0\quad . \tag{C.9}$$

Define now:

$$\vec{F}\ (\vec{\alpha};\varepsilon)\ : \mathbb{R}^3 \times \left[0,\ \varepsilon^o\right) \to \mathbb{R}^3$$

for ε^o small enough, as

$$F_i\ := P\cdot U_i\ (\vec{z}(\vec{\alpha},t;\varepsilon),t;\varepsilon) = F_i\ (\varepsilon\alpha_1,\alpha_2,\alpha_3;\varepsilon),\ i = 1,2\ ,$$

$$F_3\ := P\cdot V_2\ (\vec{z}(\vec{\alpha},t;\varepsilon),t;\varepsilon) + \alpha_1 = F_3\ (\alpha_1,\alpha_2,\alpha_3;\varepsilon)\ .$$

Then, F is continuous in ε and Lipschitz in α. At $\alpha_i = \varepsilon = 0$, we get $F_1 = F_2 = 0$ by (C.9), while we choose

$$\alpha_1^o = -P \cdot V_2 \ (0,0,0;0) \ .$$

Hence

$$F_3(\alpha_1^o,0,0;\varepsilon)\Big|_{\varepsilon=0} = 0 \ .$$

By definition, $\vec{F}$ is also differentiable at $\varepsilon = 0$. One can now check that

$$det \left\{\nabla\vec{F}\Big|_{\varepsilon=0}\right\} = J(\bar{G}) \neq 0$$

by (C.4). An application of the implicit function theorem (in some weak version) leads to the existence of $\vec{\alpha}(\varepsilon)$ for which

$$\vec{F}(\vec{\alpha}(\varepsilon);\varepsilon) = 0$$

if ε small enough, and thus to the satisfaction of (C.8). Hence, $\vec{z}(\vec{\alpha}(\varepsilon),t;\varepsilon)$ is a fixed point of (C.5) which is also is a solution of (C.2). The uniqueness follows from the uniqueness of the implicit function together with the observation that any solution of (C.2) must be a fixed point of (C.5).

QED

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Fig. 1. Sketch of averaging on the $\sqrt{\varepsilon}$-time scale, illustrated for a 2:1 resonance. One of the original T^2-tori is shaded with fine dots. The covering torus $\bar{T}^2$ is outlined in heavy dashed lines. The minimal period in the slow variable ϕ, after averaging with respect to the fast variable ψ, is indicated by heavy solid lines.

Fig. 2. Diagram of quasi-preserved solution branches. (a) Elliptic (solid circle) and hyperbolic (open circle) quasi-preserved points. The resonant manifold $\{W=0\}$ is shown as a cylinder, after averaging with respect to ψ. To each value of the energy E corresponds in this diagram two, one or no pair of quasi-preserved points. The elliptic and hyperbolic branches are separated by vertical tick marks. (b) The dynamics of flow along the quasi-preserved solution branches. Arrows indicate the direction of slow changes (on the ε^{-1} time scale) of $\bar{E}$ (see Eq. (3.2)). Point 2 is a stable elliptic point (see Theorem 3c), 3 is an unstable elliptic point, 5 is a hyperbolic (unstable) point (see Theorem 1), points 4 and 6 correspond to resonance breaking (compare Theorem 2), and 1 to resonance trapping.

Fig. 3. Projection onto a plane E = *const* of the flow near the elliptic branch at a nonstationary point, where $d\phi^P/d\tau < 0$. The light solid contours represent Hamiltonian-like oscillations around the elliptic branch, and the heavy dashed line represents the contour (5.27) on which the direction of the flow is reversed. The characteristic time scale of oscillation is $\sqrt{\varepsilon}$, the center of the oscillations changes on an ε time scale, and the amplitude and period of oscillation changes on an $\varepsilon^{3/2}$ time scale.

Fig. 4. Large-amplitude oscillations centered on an elliptic branch and bounded by the reconnected (homoclinic) separatrix at the boundary of the adjacent hyperbolic branch. (a) Projection onto the resonant manifold (compare Fig. 2); double-headed arrows indicate amplitude of oscillation. (b) Perspective view of finite-period (dashed) and infinite-period (homoclinic: solid) oscillations, after averaging out the fast phase.

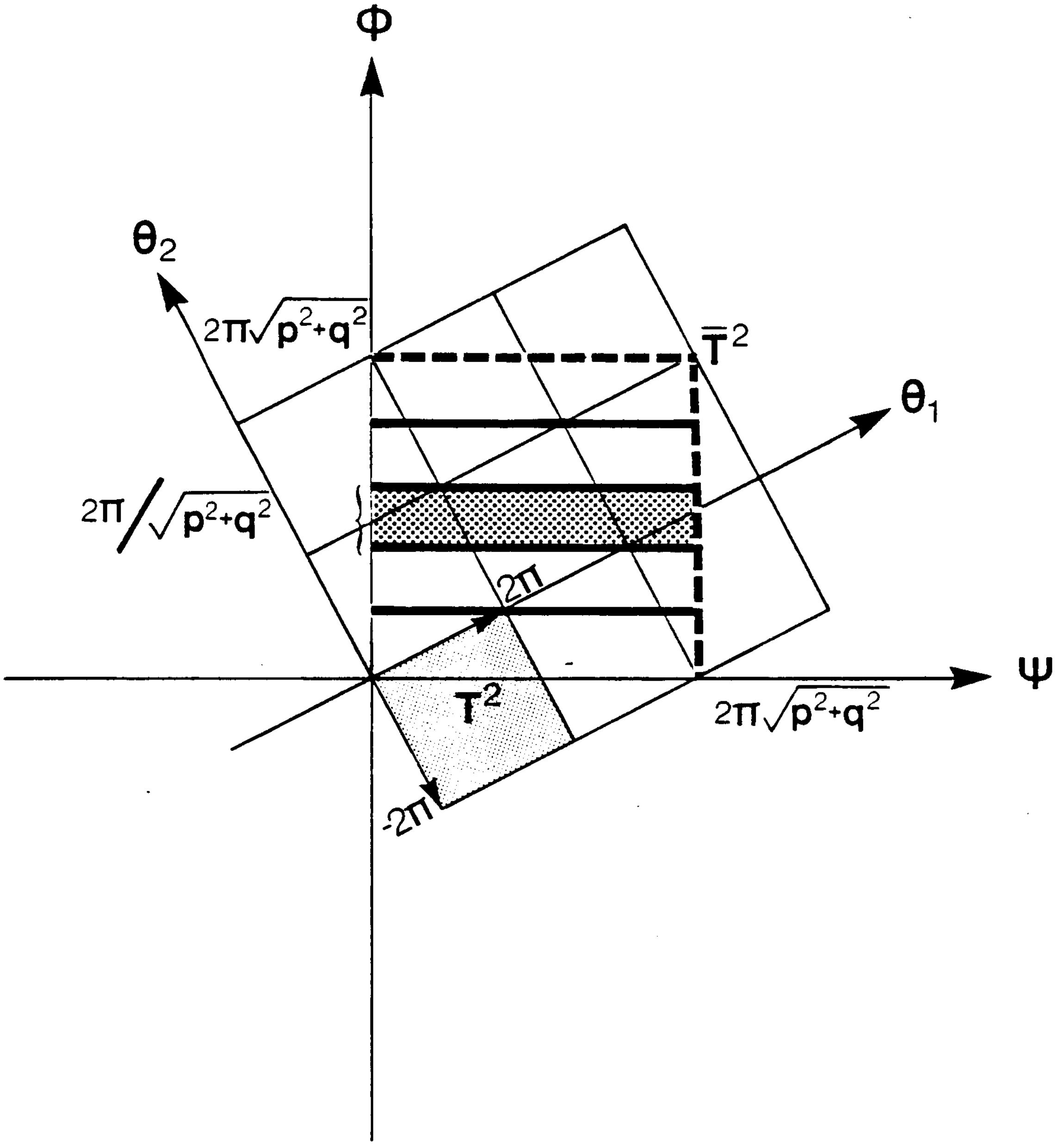

Fig. 1

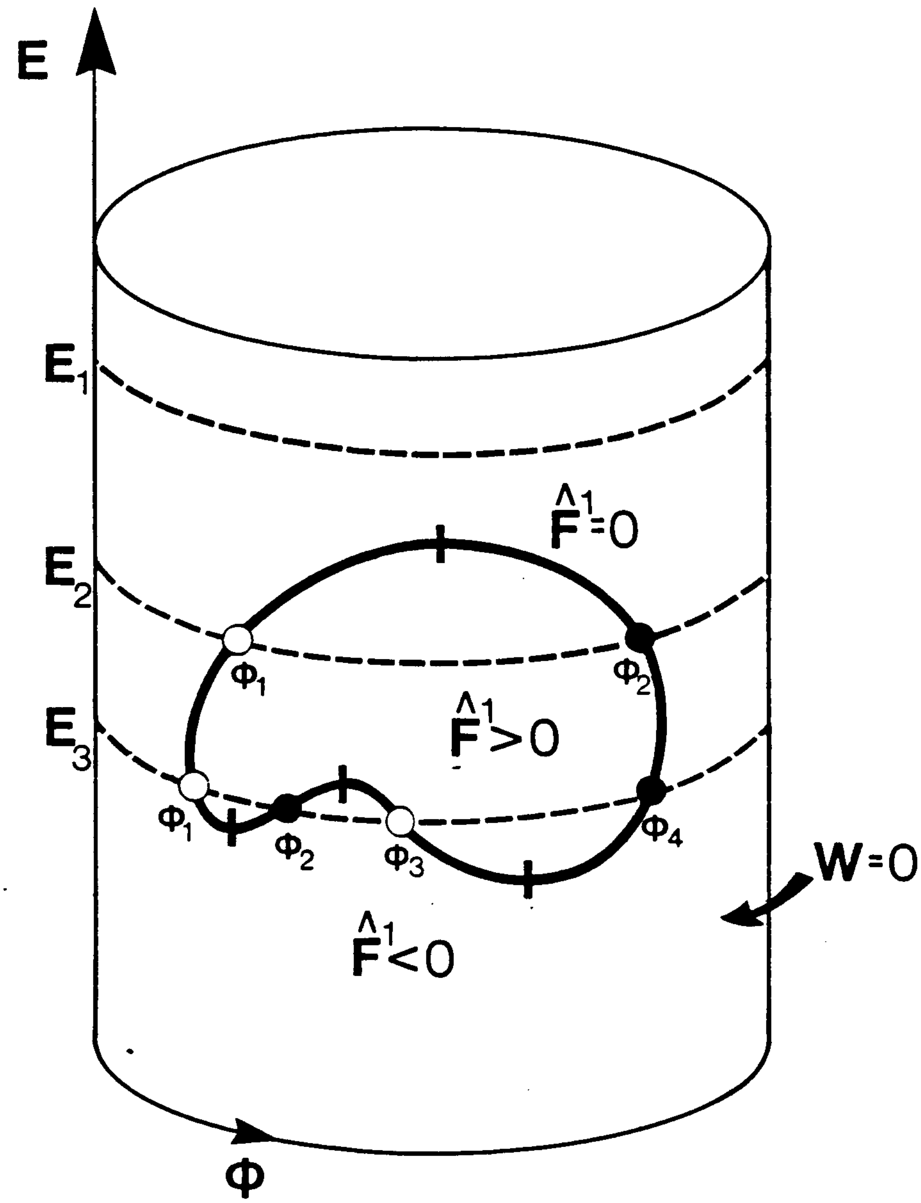

Fig. 2a

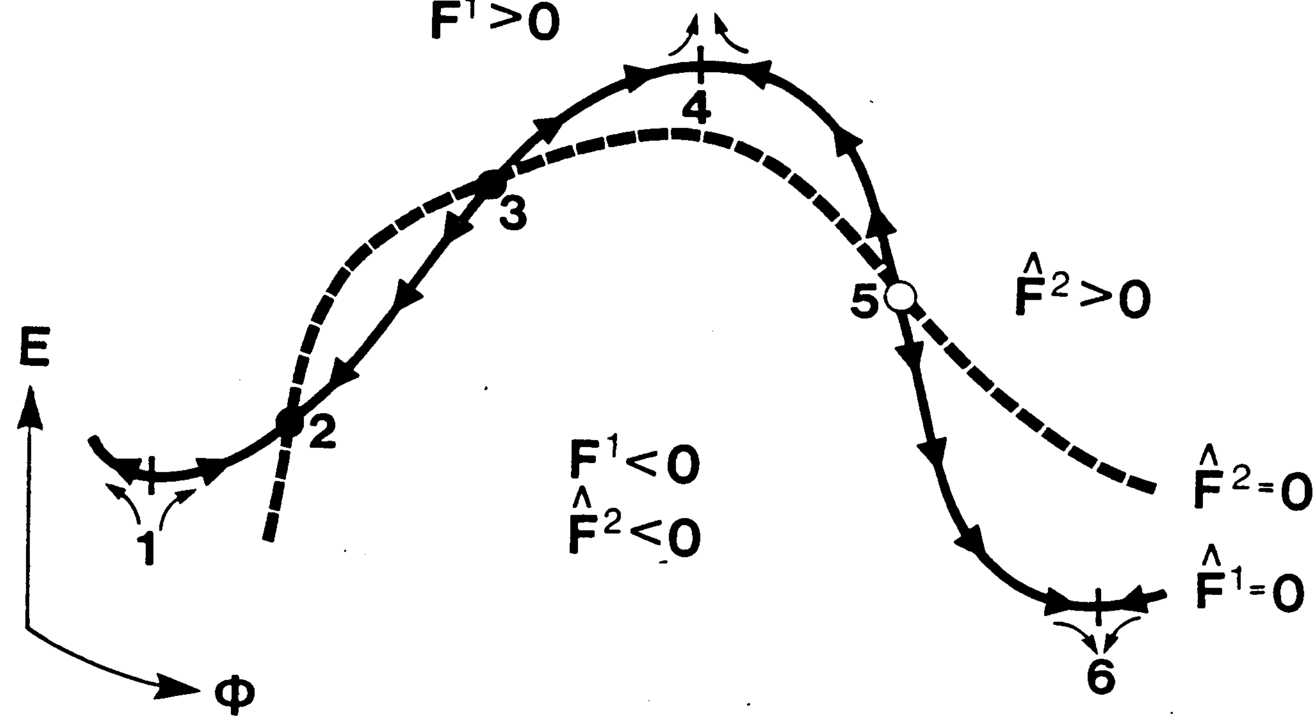

Fig. 26

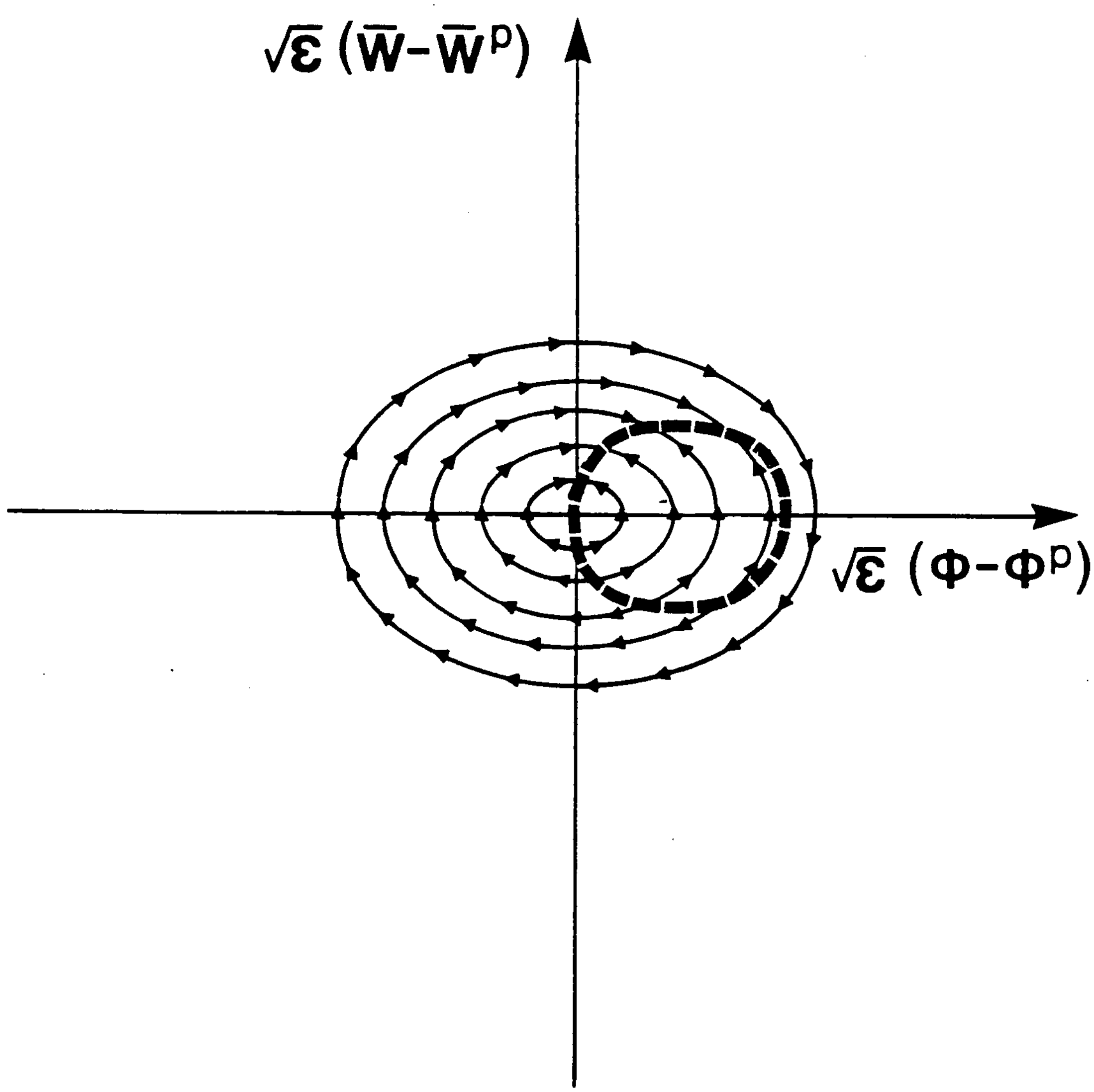

Fig. 3

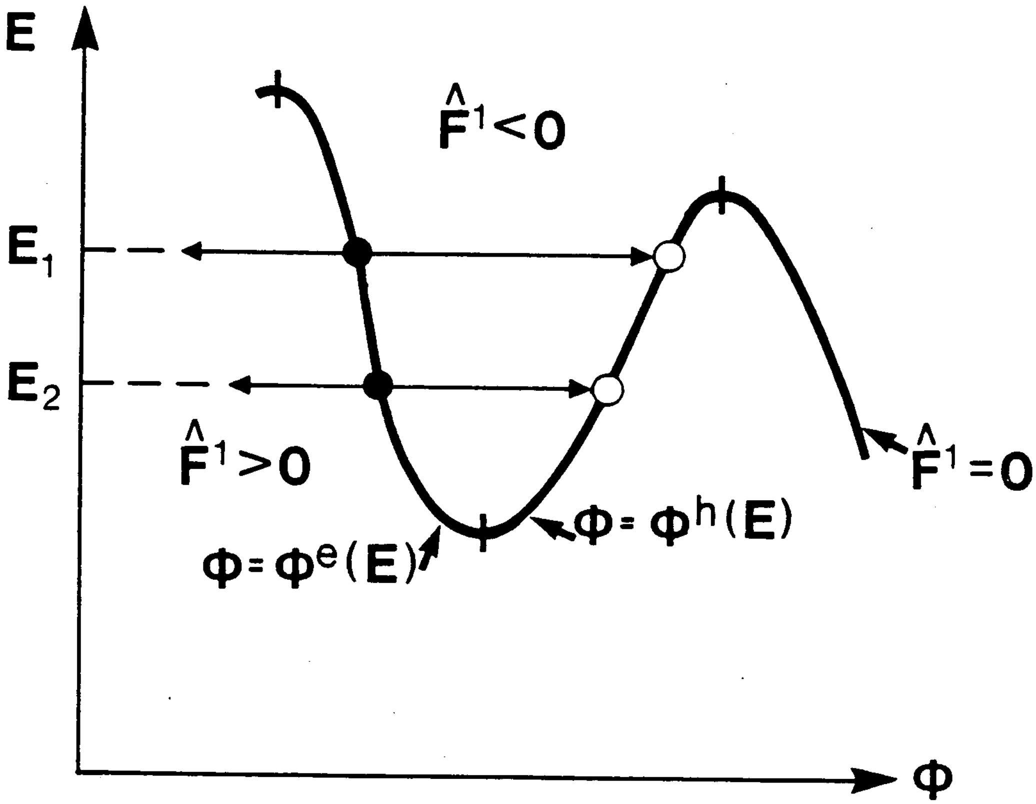

Fig. 4a

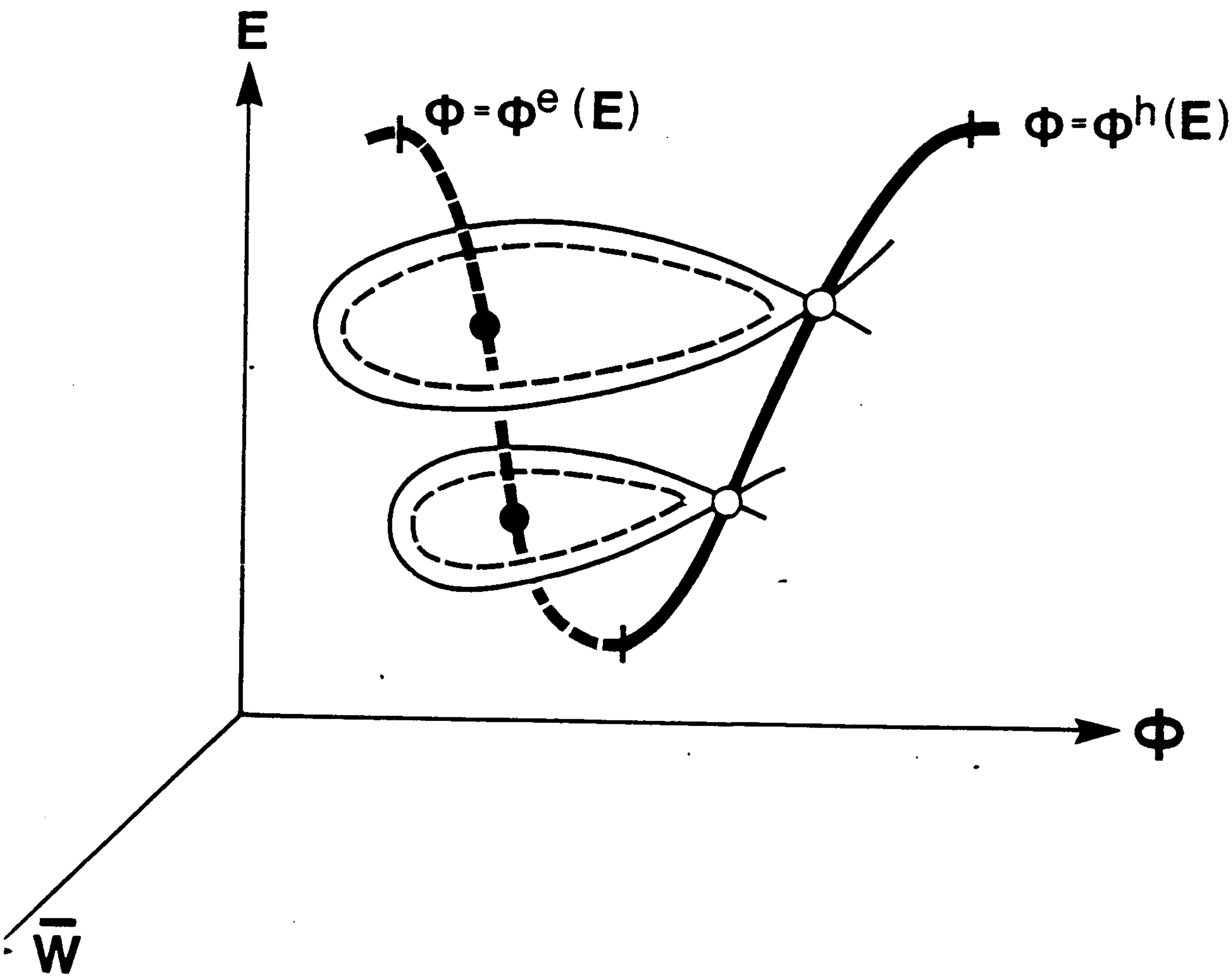

Fig. 4b